CONTINUITY OF LYAPUNOV FUNCTIONS AND OF ENERGY LEVEL FOR A GENERALIZED GRADIENT SEMIGROUP

E. R. ARAGÃO-COSTA[1], T. CARABALLO[2], A. N. CARVALHO[3], AND J. A. LANGA[4]

Abstract. The global attractor of a gradient-like semigroup has a Morse decomposition. Associated to this Morse decomposition there is a Lyapunov function (differentiable along solutions)-defined on the whole phase space- which proves relevant information on the structure of the attractor. In this paper we prove the continuity of these Lyapunov functions under perturbation. On the other hand, the attractor of a gradient-like semigroup also has an energy level decomposition which is again a Morse decomposition but with a total order between any two components. We claim that, from a dynamical point of view, this is the optimal decomposition of a global attractor; that is, if we start from the finest Morse decomposition, the energy level decomposition is the coarsest Morse decomposition that still produces a Lyapunov function which gives the same information about the structure of the attractor. We also establish sufficient conditions which ensure the stability of this kind of decomposition under perturbation. In particular, if connections between different isolated invariant sets inside the attractor remain under perturbation, we show the continuity of the energy level Morse decomposition. The class of Morse-Smale systems illustrates our results.

1. Introduction

Qualitative properties of infinite-dimensional dynamical systems has been receiving very much attention throughout the last four decades (see, for instance, [5], [9], [14] or [2]). The analysis of compact attracting invariant sets has developed a profound area of research, providing crucial information for an increasing number of models for phenomena from Physics, Biology, Economics, Engineering and others.

The asymptotic behaviour of a dissipative system can be described by a study of its associated global attractor. Moreover, a careful study of the geometrical structure -and its stability under perturbations- of the global attractor leads to the understanding of its internal dynamics, which, essentially, describes the long time behaviour of the whole system.

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The most general result in this line follows from [4], which describes any flow on a compact metric space as a decomposition of chain recurrent isolated invariant sets and connections between them. In the terminology of [4], this is called a Morse decomposition of a compact invariant set (see Definition 2.10 below), and has been considered in different frameworks, as in the case of flows ([4]) and semiflows on compact spaces ([13]), or even compact and non-compact topological spaces ([8, 11, 12]).

Recently, it has been introduced in [3] the so-called *gradient-like semigroups* with respect to a disjoint family of isolated invariant sets $\mathbf{\Xi} = (\Xi_1, \cdots, \Xi_n)$ on the global attractor (see Definition 2.8 below) in Banach spaces, as an intermediate concept between gradient semigroups (i.e., those possessing a Lyapunov function) and semigroups possessing a gradient-like attractor (that is, an attractor that is characterized as the union of the unstable sets of associated isolated invariant sets).

In [1], given a gradient-like semigroup in a general metric space, we construct a differentiable (along solutions) generalized Lyapunov function proving that gradient-like semigroups are in fact gradient semigroups. This function is not only constant on each isolated invariant set as in the classical theory of [4], but it also detects the points in the phase space with orbits having a single value of this function, a crucial property of Lyapunov functions (see, for instance, [5]). Indeed, we will say that a semigroup $\{T(t) : t \geq 0\}$ with a global attractor $\mathcal{A}$ and a disjoint family of isolated invariant sets $\mathbf{\Xi} = \{\Xi_1, \cdots, \Xi_n\}$ is a *generalized gradient semigroup* with respect to $\mathbf{\Xi}$ if there exists a continuous function $V : X \to \mathbb{R}$ such that, V is constant in each in each Ξ_i, $1 \leq i \leq n$, $[0, \infty) \ni t \mapsto V(T(t)x) \in \mathbb{R}$ is decreasing for each $x \in X$, and $V(T(t)x) = V(x)$ for all $t \geq 0$ if and only if $x \in \bigcup_{i=1}^{n} \Xi_i$. For the construction of the Lyapunov function, it is proved in [1] that the disjoint family of isolated invariant sets of a gradient-like semigroup on a general metric space can be reordered in such a way that it becomes a Morse decomposition for the global attractor. A refinement of the results from [4] leads to define a generalized Lyapunov function, not only on the attractor but on the whole phase space. In addition, the Lyapunov function $V : X \to \mathbb{R}$ of a generalized gradient-like semigroup can be chosen in such a way that $V(\Xi_j) = j$.

Moreover, as gradient-like semigroups are stable under perturbation (see [3]), we conclude that gradient semigroups are stable under perturbation. In other words, the existence of a continuous Lyapunov function is robust under perturbation. In this paper, we are able to go further in this direction, i.e., we provide conditions for which not only a perturbation of a gradient semigroup is still gradient, but also the associated Lyapunov functions move continuously under the perturbation. A careful study of the upper and lower semicontinuity of local attractors and repellers will be crucial in our argument.

On the other hand, observe that any Morse decomposition $\mathbf{\Xi} = (\Xi_1, \cdots, \Xi_n)$ of a compact invariant set $\mathcal{A}$ leads to a partial order among the isolated invariant sets Ξ_i; that is, we can define an order between two isolated invariant sets Ξ_i and Ξ_j if there is a chain of global solutions $\{\xi_\ell,\ 1 \leq \ell \leq j - i\}$, with $\lim_{t\to\infty} \xi_\ell(t) = \Xi_{i+\ell-1}$ and $\lim_{t\to-\infty} \xi_\ell(t) = \Xi_{i+\ell}$,

$1 \leq \ell \leq j-i$. This defines a partial order and some of the isolated invariant sets in Ξ may not be comparable. In Section 4 we rewrite and expand the construction in [1] of a new Morse decomposition of the attractor for a generalized gradient-like semigroup which improves the construction and dynamical properties of its associated Lyapunov function. Indeed, we show that, given any generalized gradient-like semigroup with respect to the disjoint family of isolated invariant sets $\Xi = (\Xi_1, \cdots, \Xi_n)$, there exists another Morse decomposition given by the so-called energy levels $\mathbf{N} = (\mathcal{N}_1, \mathcal{N}_2, \cdots, \mathcal{N}_p)$, $p \leq n$, which can be totally ordered by the flow. Each of the levels $\mathcal{N}_i$, $1 \leq i \leq p$ is made of a finite union of the isolated invariant sets in Ξ and $\mathbf{N}$ is totally ordered. The associated Lyapunov function takes different values in any two different sets of $\mathbf{N}$ and any two elements of Ξ which are contained in the same element of $\mathbf{N}$ (same energy level) are not connected.

Because of this energy level decomposition can be made from any gradient-like semigroup (i.e., for any Morse decomposition with a finite number of components), when we start form the finest Morse decomposition of an invariant set in the sense of [12], we claim that our new dynamical decomposition is optimal, since its associated Lyapunov function is the simplest one in order to describe connected isolated invariant sets inside the global attractor.

We recall that, given a Morse decomposition of an attractor, it can be continuous under perturbation even if the connections between sets are destroyed (see figures in Section 3). This is saying that when we describe the geometric structure of the attractor using the associated isolated invariant subsets it may, under perturbation, change drastically the way these isolated invariant subsets are connected. In Section 5 we prove that, if connections are kept under perturbation, then the energy level decomposition is stable under perturbation. There exists a general class of semigroups satisfying this last property, being Morse-Smale systems ([7], [6]) the prototype of them.

2. Morse decomposition of global attractors for generalized gradient-like semigroups

Let X be a metric space with metric $d : X \times X \to \mathbb{R}^+$, where $\mathbb{R}^+ = [0, \infty)$. Given a subset $A \subset X$, the $\epsilon-$neighborhood of A is the set $\mathcal{O}_\epsilon(A) = \{x \in X : d(x, a) < \epsilon \text{ for some } a \in A\}$

Definition 2.1. *A family of mappings $\{T(t) : t \geq 0\}$ is a semigroup in X if*

- *$T(0) = I_X$, with I_X being the identity map in X,*
- *$T(t + s) = T(t)T(s)$, for all $t, s \in \mathbb{R}^+$ and*
- *$\mathbb{R}^+ \times X \ni (t, x) \mapsto T(t)x \in X$ is continuous.*

The notion of invariance plays a fundamental role in the study of the asymptotic behavior of semigroups

Definition 2.2. *A subset A of X is said invariant under the action semigroup $\{T(t) : t \geq 0\}$ if $T(t)A = A$ for all $t \geq 0$.*

Given $A, B \subset X$, the Hausdorff semidistance between A and B is given by

$$\mathrm{dist}(A,B) = \sup_{a\in A}\inf_{b\in B} d(a,b),$$

and the Hausdorff distance by

$$\mathrm{d_H}(A,B) := \mathrm{dist}(A,B) + \mathrm{dist}(B,A).$$

For any subsets A, B and C in X it holds

$$\mathrm{dist}(A,C) \le \mathrm{dist}(A,B) + \mathrm{dist}(B,C).$$

Definition 2.3. *Given two subsets A, B of X we say that A attracts B under the action of the semigroup $\{T(t) : t \ge 0\}$ if $\mathrm{dist}(T(t)B, A) \overset{t\to\infty}{\longrightarrow} 0$ and we say that A absorbs B under the action of $\{T(t) : t \ge 0\}$ if there is a $t_B > 0$ such that $T(t)B \subset A$ for all $t \ge t_B$.*

With this we are in condition to define *global attractors*.

Definition 2.4. *A subset $\mathcal{A}$ of X is a* global attractor *for a semigroup $\{T(t) : t \ge 0\}$ if it is compact, invariant under the action of $\{T(t) : t \ge 0\}$ and for every bounded subset B of X we have that $\mathcal{A}$ attracts B under the action of $\{T(t) : t \ge 0\}$.*

2.1. **Gradient-like semigroups and Morse decomposition of attractors.** Next we seek to introduce the notion of generalized gradient-like semigroups (see [3]). To that end we first need the definition of isolated invariant set.

Definition 2.5. *Let $\{T(t) : t \ge 0\}$ be a semigroup. We say that an invariant set $\Xi \subset X$ for the semigroup $\{T(t) : t \ge 0\}$ is an isolated invariant set if there is an $\epsilon > 0$ such that Ξ is the maximal invariant subset of $\mathcal{O}_\epsilon(\Xi)$.*

A disjoint family of isolated invariant sets is a family $\{\Xi_1, \cdots, \Xi_n\}$ of isolated invariant sets with the property that, for some $\epsilon > 0$,

$$\mathcal{O}_\epsilon(\Xi_i) \cap \mathcal{O}_\epsilon(\Xi_j) = \varnothing,\ 1 \le i < j \le n.$$

Definition 2.6. *A global solution for a semigroup $\{T(t) : t \ge 0\}$ is a continuous function $\xi : \mathbb{R} \to X$ with the property that $T(t)\xi(s) = \xi(t+s)$ for all $s \in \mathbb{R}$ and for all $t \in \mathbb{R}^+$. We say that $\xi : \mathbb{R} \to X$ is a global solution through $x \in X$ if it is a global solution and $\xi(0) = x$.*

Definition 2.7. *Let $\{T(t) : t \ge 0\}$ be a semigroup which has a disjoint family of isolated invariant sets $\boldsymbol{\Xi} = \{\Xi_1, \cdots, \Xi_n\}$. A homoclinic structure associated to $\boldsymbol{\Xi}$ is a subset $\{\Xi_{k_1}, \cdots, \Xi_{k_p}\}$ of $\boldsymbol{\Xi}$ ($p \le n$) together with a set of global solutions $\{\xi_1, \cdots, \xi_p\}$ such that*

$$\Xi_{k_j} \overset{t\to-\infty}{\longleftarrow} \xi_j(t) \overset{t\to\infty}{\longrightarrow} \Xi_{k_{j+1}},\ 1 \le j \le p$$

where $\Xi_{k_{p+1}} := \Xi_{k_1}$.

We are now ready to define generalized gradient-like semigroups ([3])

Definition 2.8. *Let $\{T(t) : t \geq 0\}$ be a semigroup with a global attractor $\mathcal{A}$ and a disjoint family of isolated invariant sets $\mathbf{\Xi} = \{\Xi_1, \cdots, \Xi_n\}$. We say that $\{T(t) : t \geq 0\}$ is a generalized gradient-like semigroup associated to $\mathbf{\Xi}$ if*

- *For any global solution $\xi : \mathbb{R} \to \mathcal{A}$ there are $1 \leq i, j \leq n$ such that*
$$\Xi_i \overset{t\to-\infty}{\longleftarrow} \xi(t) \overset{t\to\infty}{\longrightarrow} \Xi_j.$$
- *There is no homoclinic structure associated to $\mathbf{\Xi}$.*

Now we will introduce the notion of a Morse decomposition for an attractor $\mathcal{A}$ of a gradient-like semigroup $\{T(t) : t \geq 0\}$. We start with the notion of attractor-repeller pairs.

Definition 2.9. *Let $\{T(t) : t \geq 0\}$ be a semigroup with a global attractor $\mathcal{A}$. We say that a non-empty subset A of $\mathcal{A}$ is a local attractor if there is an $\epsilon > 0$ such that $\omega(\mathcal{O}_\epsilon(A)) = A$. The repeller A^* associated to a local attractor A is the set defined by*
$$A^* = \{x \in \mathcal{A} : \omega(x) \cap A = \varnothing\}.$$
The pair (A, A^*) is called attractor-repeller pair for $\{T(t) : t \geq 0\}$.

Note that if A is a local attractor, then A^* is closed and invariant.

Definition 2.10. *Given an increasing family $\varnothing = A_0 \subset A_1 \subset \cdots \subset A_n = \mathcal{A}$, of local attractors, define $\Xi_j := A_j \cap A^*_{j-1}$, $1 \leq j \leq n$. The ordered n-upla $\mathbf{\Xi} := (\Xi_1, \Xi_2, \cdots, \Xi_n)$ is called a Morse decomposition on $\mathcal{A}$.*

Remark 2.11. *Observe that Ξ is a local attractor if and only if it is compact, invariant and attracts $\mathcal{O}_\epsilon(\Xi)$ for some $\epsilon > 0$. We observe that the above definition differs slightly from the usual definition since the local attractor is required to attract a neighborhood of Ξ in X and not in $\mathcal{A}$ as in* [4, 13].

The following results are proved in Aragão-Costa *et al.* [1]:

Lemma 2.12. *Let $\{T(t) : t \geq 0\}$ be a semigroup in X with a global attractor $\mathcal{A}$ and an attractor-repeller (A, A^*). A global solution $\xi : \mathbb{R} \to X$ of $\{T(t) : t \geq 0\}$ with the property that $\xi(t) \in \mathcal{O}_\delta(A^*)$ for all $t \leq 0$ for some $\delta > 0$ such that $\mathcal{O}_\delta(A^*) \cap A = \varnothing$ must satisfy $d(\xi(t), A^*) \overset{t\to-\infty}{\longrightarrow} 0$.*

Lemma 2.13. *Let $\{T(t) : t \geq 0\}$ be a semigroup in X with a global attractor $\mathcal{A}$ and (A, A^*) an attractor-repeller for $\{T(t) : t \geq 0\}$. If $\xi : \mathbb{R} \to X$ is a global bounded solution for $\{T(t) : t \geq 0\}$ through $x \notin A \cup A^*$, then $\xi(t) \overset{t\to\infty}{\longrightarrow} A$ and $\xi(t) \overset{t\to-\infty}{\longrightarrow} A^*$. Furthermore, if $x \in X \backslash \mathcal{A}$ then, $T(t)x \overset{t\to\infty}{\longrightarrow} A \cup A^*$.*

Corollary 2.14. *If $\{T(t) : t \geq 0\}$ is a semigroup in X with a global attractor $\mathcal{A}$ and (A, A^*) is an attractor-repeller pair for $\{T(t) : t \geq 0\}$, then $\{T(t) : t \geq 0\}$ is a generalized gradient-like semigroup associated to the disjoint family of isolated invariant sets $\{A, A^*\}$.*

In [1] we describe the construction of a Morse decomposition of the attractor of a gradient-like semigroup associated to the disjoint family of isolated invariant sets $\{\Xi_1, \cdots, \Xi_n\}$ and of the associated collection of increasing local attractors starting from the collection of isolated invariant sets $\{\Xi_1, \cdots, \Xi_n\}$. For the sake of completeness, we recall such a construction here.

Let $\{T(t) : t \geq 0\}$ be a generalized gradient-like semigroup with associated family of isolated invariant sets $\mathbf{\Xi} = \{\Xi_1, \cdots, \Xi_n\}$. If (after possible reordering) Ξ_1 is a local attractor for $\{T(t) : t \geq 0\}$ and

$$\Xi_1^* = \{a \in \mathcal{A} : \omega(a) \cap \Xi_1 = \varnothing\}$$

each Ξ_i, $i > 1$ is contained in Ξ_1^* and that for any $a \notin \mathcal{A} \backslash \{\Xi_1 \cup \Xi_1^*\}$ and global solution $\phi : \mathbb{R} \to \mathcal{A}$ with $\phi(0) = a$ we have that

$$\Xi_1^* \overset{t\to-\infty}{\longleftarrow} \phi_j(t) \overset{t\to\infty}{\longrightarrow} \Xi_1.$$

Considering the restriction $T_1(t)$ of $T(t)$ to Ξ_1^* we have that $T_1(t)$ is a generalized gradient-like semigroup in $\Xi_1^* =: \Xi_{1,0}$ with isolated invariant sets $\{\Xi_2, \cdots, \Xi_n\}$ and we may assume without loss of generality that Ξ_2 is a local attractor for the semigroup $\{T_1(t) : t \geq 0\}$ in Ξ_1^*. If $\Xi_{2,1}^*$ is the repeller associated to the isolated invariant set Ξ_2 for $\{T_1(t) : t \geq 0\}$ in Ξ_1^* we may proceed and consider the restriction $\{T_2(t) : t \geq 0\}$ of the semigroup $\{T_1(t) : t \geq 0\}$ to $\Xi_{2,1}^*$ and $\{T_2(t) : t \geq 0\}$ is a generalized gradient-like semigroup in $\Xi_{2,1}^*$ with associated isolated invariant sets $\{\Xi_3, \cdots, \Xi_n\}$.

Proceeding with this until all isolated invariant sets are exhausted we obtain a reordering of $\{\Xi_1, \cdots, \Xi_n\}$ in such a way that Ξ_j is a local attractor for the restriction of $\{T(t) : t \geq 0\}$ to $\Xi_{j-1,j-2}^*$ ($\Xi_{0,-1}^* := \mathcal{A}$).

Definition 2.15. *Let $\{T(t) : t \geq 0\}$ be a semigroup. The* unstable set *of an invariant set Ξ is defined by*

$$W^{\mathrm{u}}(\Xi) = \{\, z \in X : \text{ there is a global solution } \xi : \mathbb{R} \to X \text{ such that } \xi(0) = z \text{ and } \lim_{t\to-\infty} \mathrm{dist}(\xi(t), \Xi) = 0\}.$$

Define $A_0 = \varnothing$, $A_1 = \Xi_1$ and for $j = 2, 3, \cdots, n$

$$A_j = A_{j-1} \cup W^{\mathrm{u}}(\Xi_j) = \bigcup_{i=1}^{j} W^u(\Xi_i). \tag{2.1}$$

It is clear that $A_n = \mathcal{A}$.

Theorem 2.16. *(Aragão-Costa et al.* [1]*) Let $\{T(t) : t \geq 0\}$ be a generalized gradient-like semigroup with associated family of isolated invariant sets $\mathbf{\Xi} = \{\Xi_1, \cdots, \Xi_n\}$ reordered in such a way that Ξ_j is an attractor for the restriction of $\{T(t) : t \geq 0\}$ to $\Xi_{j-1,j-2}^*$. Then A_j defined in* (2.1) *is a local attractor for $\{T(t) : t \geq 0\}$ in X, and*

$$\Xi_j = A_j \cap A_{j-1}^*, \ 1 \leq j \leq n.$$

As a consequence, $\mathbf{\Xi}$ defines a Morse decomposition on $\mathcal{A}$.

2.2. **A Lyapunov function for a generalized gradient-like semigroup.** Let us now recall some definitions and results from [1].

Definition 2.17. *We say that a semigroup $\{T(t) : t \geq 0\}$ with a global attractor $\mathcal{A}$ and a disjoint family of isolated invariant sets $\mathbf{\Xi} = \{\Xi_1, \cdots, \Xi_n\}$ is a generalized gradient semigroup with respect to $\mathbf{\Xi}$ if there is a continuous function $V : X \to \mathbb{R}$ such that, V is constant in Ξ_i, for each $1 \leq i \leq n$, $[0, \infty) \ni t \mapsto V(T(t)x) \in \mathbb{R}$ is decreasing for each $x \in X$ and $V(T(t)x) = V(x)$ for all $t \geq 0$ if and only if $x \in \bigcup_{i=1}^{n} \Xi_i$. A function V with the properties above is called a Lyapunov function for the generalized gradient semigroup $\{T(t) : t \geq 0\}$ with respect to $\mathbf{\Xi}$.*

Proposition 2.18. *Let $\{T(t) : t \geq 0\}$ be a semigroup in a metric space (X, d) with global attractor $\mathcal{A}$, and let (A, A^*) be an attractor-repeller pair in $\mathcal{A}$. Then, there exists a function $f : X \to \mathbb{R}$ satisfying the following:*

- *(i) $f : X \to \mathbb{R}$ is continuous in X.*
- *(ii) $f : X \to \mathbb{R}$ is non-increasing along solutions.*
- *(iii) $f^{-1}(0) = A$ and $f^{-1}(1) \cap \mathcal{A} = A^*$.*
- *(iv) Given $z \in X$, if $f(T(t)z) = f(z)$ for all $t \geq 0$, then $z \in (A \cup A^*)$.*

Theorem 2.19. *(Aragao-Costa et al. [1]) Let $\{T(t) : t \geq 0\}$ be a semigroup with global attractor $\mathcal{A}$ and a disjoint family of isolated invariant sets $\mathbf{\Xi} = \{\Xi_1, \cdots, \Xi_n\}$. Then, $\{T(t) : t \geq 0\}$ is a generalized gradient semigroup with respect to $\mathbf{\Xi}$ if and only if it is a generalized gradient-like semigroup with respect to $\mathbf{\Xi}$. Moreover, $[0, \infty) \ni t \mapsto V(T(t)z)$ is differentiable for all $z \in X$. Finally, the Lyapunov function $V : X \to \mathbb{R}$ of a generalized gradient-like semigroup may be chosen in such a way that $V(\Xi_k) = k$, $k = 1, \cdots, n$.*

2.3. **Stability under perturbations of generalized gradient semigroups.** We introduce the notions of continuity and asymptotic compactness for a parameter dependent family. We start with the notion of continuity for a family of semigroups.

Definition 2.20. *A family of semigroups $\{T_\eta(t) : t \geq 0\}_{\eta \in [0,1]}$ is said to be continuous at $\eta = 0$ if $T_\eta(t)x \stackrel{\eta \to 0}{\longrightarrow} T_0(t)x$ uniformly for (t, x) in compact subsets of $\mathbb{R}^+ \times X$.*

Definition 2.21. *A family of semigroups $\{T_\eta(t) : t \geq 0\}_{\eta \in [0,1]}$ is said to be* collectively asymptotically compact *at $\eta = 0$ if, given a sequence $(\eta_k)_{k \in \mathbb{N}}$ with $\eta_k \stackrel{k \to \infty}{\longrightarrow} 0$, a bounded sequence $(x_k)_{k \in \mathbb{N}}$ in X and a sequence $(t_k)_{k \in \mathbb{N}}$ in $\mathbb{R}^+$ with $t_k \stackrel{k \to \infty}{\longrightarrow} \infty$, then $(T_{\eta_k}(t_k)x_k)$ is relatively compact.*

We are now ready to state the following result from [3].

Theorem 2.22 (Carvalho-Langa). *Let $\{T_\eta(t) : t \geq 0\}_{\eta \in [0,1]}$ be a collectively compact family of semigroups which is continuous at $\eta = 0$. Assume that*

a) $\{T_\eta(t) : t \geq 0\}$ *possesses a global attractor* $\mathcal{A}_\eta$ *for each* $\eta \in [0,1]$ *and* $\cup_{\eta\in[0,1]}\mathcal{A}_\eta$ *is bounded.*
b) *There exists* $n \in \mathbb{N}$ *such that* $\mathcal{A}_\eta$ *has* n *isolated invariant sets* $\boldsymbol{\Xi}_\eta = \{\Xi_{1,\eta}, \cdots, \Xi_{n,\eta}\}$ *for all* $\eta \in [0,1]$, *and* $\sup_{1\leqslant i\leqslant n} \mathrm{d_H}(\Xi_{i,\eta}, \Xi_{i,0}) \xrightarrow{\eta\to 0} 0$.
c) $\{T_0(t) : t \geq 0\}$ *is a generalized gradient-like semigroup.*

Then, there exists $\eta_0 > 0$ *such that, for all* $\eta \leqslant \eta_0$, $\{T_\eta(t) : t \geq 0\}$ *is a generalized gradient-like semigroup associated to* $\boldsymbol{\Xi}_\eta$ *and consequently*

$$\mathcal{A}_\eta = \cup_{i=1}^n W^u(\Xi_{i,\eta}), \ \forall \eta \in [0, \eta_0].$$

As an immediate consequence of this result and the ones in Section 2.2 we have the following result.

Corollary 2.23. *Under the assumption of Theorem 2.22, there exists* $\eta_0 > 0$ *such that, for all* $\eta \leqslant \eta_0$, $\{T_\eta(t) : t \geq 0\}$ *is a generalized gradient semigroup.*

Corollary 2.24. *Under the assumption of Theorem 2.22, suppose there exists* $n \in \mathbb{N}$ *such that* $\mathcal{A}_\eta$ *has* n *stationary solutions* $\mathcal{S}_\eta = \{\xi_{1,\eta}, \cdots, \xi_{n,\eta}\}$ *for all* $\eta \in [0,1]$ *and* $\sup_{1\leqslant i\leqslant n} d(\xi_{i,\eta}, \xi_{i,0}) \xrightarrow{\eta\to 0} 0$. *Then, there exists* $\eta_0 > 0$ *such that, for all* $\eta \leqslant \eta_0$, $\{T_\eta(t) : t \geq 0\}$ *is a gradient semigroup in the sense of* [5].

Remark 2.25. *The previous theorem supposes the continuity of the isolated invariant sets in order to prove the stability of the generalized gradient-like semigroups under perturbation. Note (cf.* [3]*) that from a perturbation of a gradient-like semigroup it could emerge a gradient-like semigroup with a different collection of isolated invariant sets.*

On the other hand, for a generalized gradient-like semigroup, even when the isolated sets behave continuously under perturbation, some of the connections between them may change. So the dynamics under perturbation could suffer drastic changes. This fact allows that the Lyapunov functions that we have constructed behave discontinuously under perturbation.

3. Continuity of the Lyapunov function under perturbation

Now, we will analyze the continuity of the Lyapunov function under suitable perturbations.

Definition 3.1. *Let* $(A_\eta)_{\eta\in[0,1]}$ *be a family of sets in a metric space* X *with distance* $d : X \times X \to \mathbb{R}^+$. *We say that this family is upper semicontinuous (u.s.c.) at* $\eta = 0$ *if*

$$\lim_{\eta\to 0^+} \mathrm{dist}\,(A_\eta, A_0) = 0.$$

We say that this family is lower semicontinuous (l.s.c.) at $\eta = 0$ *if*

$$\lim_{\eta\to 0^+} \mathrm{dist}\,(A_0, A_\eta) = 0.$$

Finally, the family is said to be continuous at $\eta = 0$ if it is upper and lower semicontinuous, i.e., when it holds

$$\lim_{\eta \to 0^+} \mathrm{d_H}\left(A_\eta, A_0\right) = 0.$$

Lemma 3.2. *Let $\{T_\eta(t) : t \geq 0\}_{\eta \in [0,1]}$ be a family of collectively asymptotically compact semigroups in a metric space X which is continuous at $\eta = 0$ (see Definition 2.20). Assume that each $\{T_\eta(t) : t \geq 0\}$ has a global attractor $\mathcal{A}_\eta$ and that $\bigcup_{\eta \in [0,1]} \mathcal{A}_\eta$ is bounded in X. Let $(A_\eta)_{\eta \in [0,1]}$ be a family of subsets in X such that $A_\eta \subset \mathcal{A}_\eta$ and A_0 is a local attractor for $\{T_0(t) : t \geq 0\}$ with $\omega\left(\mathcal{O}_\varepsilon(A_0)\right) = A_0$, for some $\varepsilon > 0$.*

Then, if $(A_\eta)_{\eta \in [0,1]}$ is continuous at $\eta = 0$, given $\delta \in (0, \varepsilon)$ there exist $\delta' \in (0, \delta)$ and $\eta_0 > 0$ such that for all $\eta \in [0, \eta_0]$ it holds

$$\gamma_\eta^+\left(\mathcal{O}_{\delta'}(A_\eta)\right) \subset \mathcal{O}_\delta(A_\eta),$$

where $\gamma_\eta^+\left(\mathcal{O}_{\delta'}(A_\eta)\right)$ denotes the positive orbit of the set $\mathcal{O}_{\delta'}(A_\eta)$ associated to $\{T_\eta(t) : t \geq 0\}$.

Proof. Suppose not, then there exist $\delta \in (0, \epsilon)$ and sequences $(z_j)_{j \in \mathbb{N}}$ in X, $(\eta_j)_{j \in \mathbb{N}}$ in $[0,1]$ and $(t_j)_{j \in \mathbb{N}}$ in $\mathbb{R}$ such that $\eta_j \overset{j \to \infty}{\to} 0^+$, $t_j \overset{j \to \infty}{\to} \infty$, $\operatorname{dist}\left(z_j, A_{\eta_j}\right) < \frac{1}{j}$ for all j,

$$\operatorname{dist}\left(T_{\eta_j}(t) z_j, A_{\eta_j}\right) < \delta \text{ for all } t \in [0, t_j) \text{ and all } j \in \mathbb{N}$$

and

$$\operatorname{dist}\left(T_{\eta_j}(t_j) z_j, A_{\eta_j}\right) = \delta \text{ for all } j \in \mathbb{N}.$$

If, for each j, we now define $\xi_j : [-t_j, \infty) \to X$ by $\xi_j(t) := T_{\eta_j}(t + t_j) z_j$, then, by the collective asymptotic compactness and the uniform convergence in compact sets, it is not difficult to see that there exist a bounded global solution $\xi_0 : \mathbb{R} \to X$ for $\{T_0(t) : t \geq 0\}$ and a subsequence for $(\xi_j)_{j \in \mathbb{N}}$, denoted the same, such that for all t, $\xi_0(t) = \lim_{j \to \infty} \xi_j(t)$.

On the other hand, given $t < 0$, for all j big enough it holds

$$\operatorname{dist}\left(\xi_j(t), A_0\right) \leq \operatorname{dist}\left(\xi_j(t), A_{\eta_j}\right) + \operatorname{dist}\left(A_{\eta_j}, A_0\right),$$

from where, by the u.s.c. of $(A_\eta)_{\eta \in [0,1]}$, we obtain that for all $t < 0$

$$\operatorname{dist}\left(\xi_0(t), A_0\right) \leq \delta,$$

and from $\delta = \operatorname{dist}\left(\xi_j(0), A_{\eta_j}\right) \leq \operatorname{dist}\left(\xi_j(0), A_0\right) + \operatorname{dist}\left(A_0, A_{\eta_j}\right)$, by the l.s.c. of $(A_\eta)_{\eta \in [0,1]}$, it follows that $\operatorname{dist}\left(\xi_0(0), A_0\right) = \delta$.

But, as $\delta < \varepsilon$, then A_0 attracts $K = \{\xi_0(t) : t \leq 0\}$, which contradicts the fact that $\operatorname{dist}\left(\xi_0(0), A_0\right) = \delta$. □

We also have the following lemma:

Lemma 3.3. *Let $\{T_\eta(t) : t \geq 0\}_{\eta\in[0,1]}$ be a family of collectively asymptotically compact semigroups in the metric space X which is continuous at $\eta = 0$. Assume that each $\{T_\eta(t) : t \geq 0\}$ possesses a global attractor $\mathcal{A}_\eta$, that $\bigcup_{\eta\in[0,1]} \mathcal{A}_\eta$ is relatively compact in X and that $\lim_{\eta\to 0} \operatorname{dist}(\mathcal{A}_\eta, \mathcal{A}) = 0$. Let also $(A_\eta)_{\eta\in[0,1]}$ be subsets of X such that each A_η is a local attractor for $\{T_\eta(t) : t \geq 0\}$ and let $\left(A^*_\eta\right)_{\eta\in[0,1]}$ be the family of associated repellers.*

Suppose there exist a positive number μ and an index $\tilde{\eta} > 0$ such that

$$\inf_{a_\eta\in A_\eta} \inf_{a^*_\eta\in A^*_\eta} d(a_\eta, a^*_\eta) \geq \mu$$

if $\eta \in [0, \tilde{\eta}]$. If $(A_\eta)_{\eta\in[0,1]}$ is lower semicontinuous at $\eta = 0$, then $\left(A^*_\eta\right)_{\eta\in[0,1]}$ is upper semicontinuous at $\eta = 0$.

Proof. Suppose that $\lim_{\eta\to 0^+} \operatorname{dist}(A_0, A_\eta) = 0$, but it is not true that $\lim_{\eta\to 0^+} \operatorname{dist}\left(A^*_\eta, A^*_0\right) = 0$. Then, there exist $\varepsilon > 0$ and a sequence $(\eta_j)_{j\in\mathbb{N}}$ in $[0, 1]$ with $\eta_j \overset{j\to\infty}{\to} 0^+$ such that

$$\operatorname{dist}\left(A^*_{\eta_j}, A^*_0\right) \geq \varepsilon \text{ for all } j \in \mathbb{N}.$$

Thus, there exists a sequence $(z_j)_{j\in\mathbb{N}}$ in X with $z_j \in A^*_{\eta_j} \subset \mathcal{A}_{\eta_j}$ and $\operatorname{dist}(z_j, A^*_0) > \frac{\varepsilon}{2}$ for all j. By the upper semicontinuity of the global attractors, we can suppose that $z_j \overset{j\to\infty}{\to} z_0$ for some $z_0 \in \mathcal{A}_0$, so that we have $\operatorname{dist}(z_0, A^*_0) \geq \frac{\varepsilon}{2}$, and, therefore, $\omega(z_0) \subset A_0$.

On the one hand, for $0 < \delta < \frac{\mu}{2}$, by Lemma 3.2, choose $\delta' \in (0, \delta)$ and $\eta_0 \in (0, \tilde{\eta}]$ such that

$$\gamma^+_\eta\left(\mathcal{O}_{\delta'}(A_\eta)\right) \subset \mathcal{O}_\delta(A_\eta), \tag{3.1}$$

if $\eta \in [0, \eta_0]$.

As $\omega(z_0) \subset A_0$, there exists $t_0 > 0$ such that $T_0(t_0) z_0 \in \mathcal{O}_{\frac{\delta'}{3}}(A_0)$.

On the other hand, by the lower semicontinuity of $(A_\eta)_{\eta\in[0,1]}$ we get the existence of $\eta_1 \in (0, \eta_0]$ such that for all $\eta \in [0, \eta_1]$ it holds

$$A_0 \subset \mathcal{O}_{\frac{\delta'}{2}}(A_\eta). \tag{3.2}$$

But, by the continuity of $\{T_\eta(t) : t \geq 0\}_{\eta\in[0,1]}$ at $\eta = 0$, there exists $\eta_2 \in (0, \eta_1]$ for which $T_{\eta_j}(t_0) z_j \in \mathcal{O}_{\frac{\delta'}{2}}(A_0)$ for all j with $\eta_j \in [0, \eta_2]$. Now, by (3.2) it holds that $\mathcal{O}_{\frac{\delta'}{2}}(A_0) \subset \mathcal{O}_{\delta'}(A_\eta)$ if $\eta \in [0, \eta_2]$, and so $T_{\eta_j}(t_0) z_j \in \mathcal{O}_{\delta'}\left(A_{\eta_j}\right)$ if $\eta_j \in [0, \eta_2]$.

Now (3.1) implies that if $\eta_j \in [0, \eta_2]$ we have that $\gamma^+_{\eta_j}\left(T_{\eta_j}(t_0) z_j\right) \subset \mathcal{O}_\delta\left(A_{\eta_j}\right)$, and so $\omega_{\eta_j}(z_j) \subset \overline{\mathcal{O}_\delta\left(A_{\eta_j}\right)}$ when $\eta_j \in [0, \eta_2]$, but, by the invariance of $A^*_{\eta_j}$ for $\left\{T_{\eta_j}(t) : t \geq 0\right\}$, it holds that $\omega_{\eta_j}(z_j) \subset A^*_{\eta_j}$ which is a contradiction as $\delta < \frac{\mu}{2}$. □

The main result of this section is the following:

Proposition 3.4. *Let $\{T_\eta(t) : t \geq 0\}_{\eta\in[0,1]}$ be a family of asymptotically collectively compact semigroups in a metric space X which is continuous at $\eta = 0$ and such that, for each $\eta \in [0,1]$, $\{T_\eta(t) : t \geq 0\}$, it has a global attractor $\mathcal{A}_\eta$. Suppose $\bigcup_{\eta\in[0,1]} \mathcal{A}_\eta$ is relatively compact in X. Let $(A_\eta)_{\eta\in[0,1]}$ be a family of local attractors for $\{T_\eta(t) : t \geq 0\}$ in X and $\left(A^*_\eta\right)_{\eta\in[0,1]}$ the associated family of repellers.*

Suppose that the family of local attractors $(A_\eta)_{\eta\in[0,1]}$, the corresponding family of repellers $\left(A^*_\eta\right)_{\eta\in[0,1]}$, and the family of global attractors $(\mathcal{A}_\eta)_{\eta\in[0,1]}$ are continuous at $\eta = 0$.

Finally, for each $\eta \in [0,1]$, let $f_\eta : X \to \mathbb{R}$ be the Lyapunov function associated to the pair $\left(A_\eta, A^*_\eta\right)$ which is defined by

$$f_\eta(z) := h_\eta(z) + k_\eta(z),$$

where, for each $z \in X$

$$h_\eta(z) := \sup_{t\geq 0} \operatorname{dist}(T_\eta(t)\, z, \mathcal{A}_\eta) \ \text{ and,}$$

$$k_\eta(z) := \sup_{t\geq 0} l_\eta(T_\eta(t)\, z) \ \text{ with } l_\eta(z) := \frac{\operatorname{dist}(z, A_\eta)}{\operatorname{dist}(z, A_\eta) + \operatorname{dist}\left(z, A^*_\eta\right)} .$$

Then, $f_\eta \overset{\eta\to 0^+}{\to} f_0$ uniformly in compact sets of de X.

Proof. We split the proof into three steps:

Step 1: $l_\eta \overset{\eta\to 0^+}{\to} l_0$ uniformly in X.

Indeed, by the triangle inequality for the Hausdorff semidistance we have that, for all $\eta \in [0,1]$ and all $z \in X$, it holds

$$|\operatorname{dist}(z, A_\eta) - \operatorname{dist}(z, A_0)| \leq \mathrm{d_H}(A_\eta, A_0)$$

and

$$\left|\operatorname{dist}\left(z, A^*_\eta\right) - \operatorname{dist}(z, A^*_0)\right| \leq \mathrm{d_H}\left(A^*_\eta, A^*_0\right).$$

Now, given $\eta \in [0,1]$ and $z \in X$ we have

$$\begin{aligned} l_\eta(z) - l_0(z) &= \frac{\operatorname{dist}(z, A_\eta)}{\operatorname{dist}(z, A_\eta) + \operatorname{dist}\left(z, A^*_\eta\right)} - \frac{\operatorname{dist}(z, A_0)}{\operatorname{dist}(z, A_0) + \operatorname{dist}(z, A^*_0)} \\ &= \frac{\operatorname{dist}(z, A_\eta)\operatorname{dist}(z, A^*_0) - \operatorname{dist}(z, A_0)\operatorname{dist}\left(z, A^*_\eta\right)}{\left[\operatorname{dist}(z, A_\eta) + \operatorname{dist}\left(z, A^*_\eta\right)\right]\left[\operatorname{dist}(z, A_0) + \operatorname{dist}(z, A^*_0)\right]}, \end{aligned}$$

and now, by adding and substracting $\operatorname{dist}(z, A_0)\operatorname{dist}(z, A^*_0)$,

$$l_\eta(z) - l_0(z) = \frac{\left[\operatorname{dist}(z, A_\eta) - \operatorname{dist}(z, A_0)\right]\operatorname{dist}(z, A^*_0) + \operatorname{dist}(z, A_0)\left[d(z, A^*_0) - \operatorname{dist}\left(z, A^*_\eta\right)\right]}{\left[\operatorname{dist}(z, A_\eta) + \operatorname{dist}\left(z, A^*_\eta\right)\right]\left[\operatorname{dist}(z, A_0) + \operatorname{dist}(z, A^*_0)\right]}.$$

Since $\inf_{a_0\in A_0} \inf_{a^*_0\in A^*_0} d(a_0, a^*_0) > \mu$ for some $\mu > 0$, the continuity of the families of local attractors and their corresponding repellers ensure the existence of $\tilde{\eta} \in (0,1]$ such that, for all $\eta \in [0,\tilde{\eta}]$, it follows that $\operatorname{dist}\left(A_\eta, A_\eta^*\right) > \mu$. Consequently, for all $z \in X$ and $\eta \in [0,\tilde{\eta}]$ we have

$$\begin{aligned}|l_\eta(z) - l_0(z)| &\le \frac{1}{\operatorname{dist}(z, A_\eta) + \operatorname{dist}\left(z, A_\eta^*\right)}\left[\mathrm{d_H}(A_\eta, A_0) + \mathrm{d_H}\left(A_\eta^*, A_0^*\right)\right] \\ &\le \frac{1}{\mu}\left[\mathrm{d_H}(A_\eta, A_0) + \mathrm{d_H}\left(A_\eta^*, A_0^*\right)\right],\end{aligned}$$

so that, for all $z \in X$ and all $\eta \in [0,\tilde{\eta}]$ we have

$$|l_\eta(z) - l_0(z)| \le \frac{1}{\mu}\left[\mathrm{d_H}(A_\eta, A_0) + \mathrm{d_H}\left(A_\eta^*, A_0^*\right)\right]$$

and hence, we obtain the uniform convergence (in X) of $l_\eta \overset{\eta\to 0^+}{\to} l_0$, from the continuity of the local attractors and their associated repellers.

Step 2: $k_\eta \overset{\eta\to 0^+}{\to} k_0$ uniformly in compact sets of X.

Given $z \in X$ consider the following three cases

Case 1: $T_0(t)\,z \overset{t\to\infty}{\to} A_0$ with $l_0(z) > 0$.

Choose $0 < \theta < \theta^+ < l_0(z)$. By the continuity of $l_0 : X \to \mathbb{R}$, let $\sigma_1 > 0$ such that $l_0(\mathcal{O}_{\sigma_1}(z)) \subset (\theta^+, 1]$ and, by Step 1, $\eta_0 \in (0,1]$ such that $l_\eta(\mathcal{O}_{\sigma_1}(z)) \subset (\theta, 1]$ for all $\eta \in [0,\eta_0]$.

On the one hand, by the continuity of $l_0 : X \to \mathbb{R}$, given $0 < \alpha < \frac{\theta}{2}$, let $\delta > 0$ such that $l_0(\mathcal{O}_\delta(A_0)) \subset [0,\alpha)$.

On the other hand, by Lemma 3.2, let $\delta' \in \left(0, \frac{\delta}{2}\right)$ and $\eta_1 \in (0,\eta_0]$ such that for each $\eta \in [0,\eta_1]$ we have

$$\gamma_\eta^+(\mathcal{O}_{\delta'}(A_\eta)) \subset \mathcal{O}_{\frac{\delta}{2}}(A_\eta). \tag{3.3}$$

Now, by the lower semicontinuity of $(A_\eta)_{\eta\in[0,1]}$ at $\eta = 0$, let $\eta_2 \in (0,\eta_1]$ such that for each $\eta \in [0,\eta_2]$ it holds

$$A_0 \subset \mathcal{O}_{\frac{\delta'}{2}}(A_\eta). \tag{3.4}$$

From the fact that $T_0(t)\,z \overset{t\to\infty}{\to} A_0$, let also $t_0 > 0$ such that $T_0(t_0)\,z \in \mathcal{O}_{\frac{\delta'}{4}}(A_0)$ and by the continuity of $T_0(t_0) : X \to X$ choose $\sigma_2 \in (0,\sigma_1]$ such that $T_0(t_0)(\mathcal{O}_{\sigma_2}(z)) \subset \mathcal{O}_{\frac{\delta'}{4}}(A_0)$. From the continuity of $\{T_\eta(t) : t \ge 0\}_{\eta\in[0,1]}$ at $\eta = 0$, we can find $\sigma_3 \in (0,\sigma_2]$ and $\eta_3 \in (0,\eta_2]$ such that for all $\eta \in [0,\eta_3]$ we have $T_\eta(t_0)(\mathcal{O}_{\sigma_3}(z)) \subset \mathcal{O}_{\frac{\delta'}{2}}(A_0)$, from where, by (3.4), we obtain that $T_\eta(t_0)(\mathcal{O}_{\sigma_3}(z)) \subset \mathcal{O}_{\delta'}(A_\eta)$ if $\eta \in [0,\eta_3]$, and from (3.3) we conclude that

$$\gamma_\eta^+(T_\eta(t_0)(\mathcal{O}_{\sigma_3}(z))) \subset \mathcal{O}_{\frac{\delta}{2}}(A_\eta) \text{ for all } \eta \in [0,\eta_3]. \tag{3.5}$$

Now observe that, from the uniform convergence of $l_\eta \overset{\eta\to 0^+}{\to} l_0$ in X, we obtain $\eta_4 \in (0,\eta_3]$ so that, for each $\eta \in [0,\eta_4]$, it holds $l_\eta(\mathcal{O}_\delta(A_0)) \subset [0,2\alpha)$, and from the upper semicontinuity of $(A_\eta)_{\eta\in[0,1]}$ in $\eta = 0$ we deduce the existence of $\eta_5 \in (0,\eta_4]$ such that, if $\eta \in [0,\eta_5]$, then $A_\eta \subset \mathcal{O}_{\frac{\delta}{2}}(A_0)$, and, therefore, $\mathcal{O}_{\frac{\delta}{2}}(A_\eta) \subset \mathcal{O}_\delta(A_0)$ for all $\eta \in [0,\eta_5]$, so that $l_\eta\left(\mathcal{O}_{\frac{\delta}{2}}(A_\eta)\right) \subset$

$[0,2\alpha)$ for all $\eta \in [0,\eta_5]$. Thus, from (3.5) we have that for each $\eta \in [0,\eta_5]$ and each $w \in \mathcal{O}_{\sigma_3}(z) \subset \mathcal{O}_{\sigma_1}(z)$ it holds

$$\sup_{t \geq t_0} l_\eta (T_\eta (t) w) \leq 2\alpha < \theta < l_\eta (w) \leq k_\eta (w),$$

so that $k_\eta(w) = \sup_{0 \leq t \leq t_0} l_\eta (T_\eta (t) w)$ for all $\eta \in [0,\eta_5]$ and all $w \in \mathcal{O}_{\sigma_3}(z)$.

Finally, given $\varepsilon > 0$, by the conclusion in Step 1, there exists $\eta_6 \in (0,\eta_5]$ such that for all $w \in X$

$$|l_\eta (w) - l_0 (w)| < \frac{\varepsilon}{2} \text{ for all } \eta \in [0,\eta_6].$$

by the uniform continuity of the function $l_0 : X \to \mathbb{R}$, consider $\beta > 0$ such that if $w, w' \in X$ satisfy $d(w,w') < \beta$ then $|l_0(w) - l_0(w')| < \frac{\varepsilon}{2}$ so that, from the continuity of $\{T_\eta(t) : t \geq 0\}_{\eta \in [0,1]}$ at $\eta = 0$, we can choose $\eta' \in (0,\eta_6]$ and $\sigma_4 \in (0,\sigma_3]$ such that

$$\sup_{w \in \mathcal{O}_{\sigma_4}(z)} \sup_{0 \leq t \leq t_0} d(T_\eta (t) w, T_0 (t) w) < \beta.$$

Thus, for all $w \in \mathcal{O}_{\sigma_4}(z)$, all $t \in [0,t_0]$ and all $\eta \in [0,\eta']$

$$|l_\eta (T_\eta (t) w) - l_0 (T_0 (t) w)| \leq |l_\eta (T_\eta (t) w) - l_0 (T_\eta (t) w)| + |l_0 (T_\eta (t) w) - l_0 (T_0 (t) w)| < \varepsilon,$$

from which

$$\sup_{w \in \mathcal{O}_{\sigma_4}(z)} |k_\eta (w) - k_0 (w)| \leq \varepsilon \text{ for all } \eta \in [0,\eta'], \tag{3.6}$$

where $\sigma_4 > 0$ and $\eta' > 0$ depends only on $z \in X$ and $\varepsilon > 0$.

Case 2: $l_0(z) = 0$.

Under these conditions, note that $z \in A_0$ and, consequently, $k_0(z) = 0$.

Given $\varepsilon > 0$, by the continuity of $l_0 : X \to \mathbb{R}$, take $\delta > 0$ such that $l_0(\mathcal{O}_\delta(A_0)) \subset [0, \frac{\varepsilon}{4})$.

Now, the uniform convergence of $(l_\eta)_{\eta \in [0,1]}$ to l_0 in X implies the existence of $\eta_0 \in (0,1]$ such that

$$l_\eta (\mathcal{O}_\delta (A_0)) \subset [0, \frac{\varepsilon}{2}) \text{ for each } \eta \in [0,\eta_0]. \tag{3.7}$$

By the upper semicontinuity of $(A_\eta)_{\eta \in [0,1]}$ at $\eta = 0$ we have the existence of $\eta_1 \in (0,\eta_0]$ such that for all $\eta \in [0,\eta_1]$ we have $A_\eta \subset \mathcal{O}_{\frac{\delta}{2}}(A_0)$, from which $\mathcal{O}_{\frac{\delta}{2}}(A_\eta) \subset \mathcal{O}_\delta(A_0)$ if $\eta \in [0,\eta_1]$. And from (3.7) we conclude that for all $\eta \in [0,\eta_1]$

$$l_\eta \left(\mathcal{O}_{\frac{\delta}{2}} (A_\eta) \right) \subset [0, \frac{\varepsilon}{2}). \tag{3.8}$$

Let also $\eta_2 \in (0,\eta_1]$ and $\delta' \in (0, \frac{\delta}{2})$, by Lemma 3.2, such that

$$\gamma_\eta^+ (\mathcal{O}_{\delta'} (A_\eta)) \subset \mathcal{O}_{\frac{\delta}{2}} (A_\eta) \text{ for all } \eta \in [0,\eta_2]. \tag{3.9}$$

Finally, consider the lower semicontinuity of $(A_\eta)_{\eta \in [0,1]}$ at $\eta = 0$. Let $\eta_3 \in (0,\eta_2]$ such that

$$A_0 \subset \mathcal{O}_{\frac{\delta'}{2}} (A_\eta) \text{ for all } \eta \in [0,\eta_3]. \tag{3.10}$$

Thus, (3.10) holds and, from (3.9), for $\eta \in [0,\eta_3]$, for all $t \geq 0$ and all $w \in \mathcal{O}_{\frac{\delta'}{2}}(A_0) \subset \mathcal{O}_{\delta'}(A_\eta)$ that $T_\eta(t)\,w \in \mathcal{O}_{\frac{\delta}{2}}(A_\eta)$ and by (3.8), we obtain that for all $\eta \in [0,\eta_3]$ and all $w \in \mathcal{O}_{\frac{\delta'}{2}}(A_0)$ it holds

$$k_\eta(w) = \sup_{t\geq 0} l_\eta(T_\eta(t)\,w) \leq \frac{\varepsilon}{2},$$

so that, in particular,

$$\sup_{w\in\mathcal{O}_{\frac{\delta'}{2}}(A_0)} |k_\eta(w) - k_0(w)| \leq \varepsilon \text{ for all } \eta \in [0,\eta_3], \tag{3.11}$$

where $\delta' > 0$ and $\eta_3 > 0$ that depends only on $\varepsilon > 0$ and A_0.

Case 3: $T_0(t)\,z \overset{t\to\infty}{\to} A_0^*$.

In this case $k_0(z) = 1$. By the continuity of $l_0 : X \to \mathbb{R}$, given $\varepsilon > 0$, let $\delta > 0$ such that

$$l_0(\mathcal{O}_\delta(A_0^*)) \subset (1 - \frac{\varepsilon}{4}, 1]$$

and, by the uniform convergence $l_\eta \overset{\eta\to 0^+}{\to} l_0$ in X, take $\eta_0 \in (0,1]$ such that

$$l_\eta(\mathcal{O}_\delta(A_0^*)) \subset (1 - \frac{\varepsilon}{2}, 1] \text{ for all } \eta \in [0,\eta_0]. \tag{3.12}$$

On the other hand, consider $t_0 > 0$ such that $T_0(t_0)\,z \in \mathcal{O}_{\frac{\delta}{2}}(A_0^*)$ and, by the continuity of $T_0(t_0) : X \to X$, take $\sigma_1 > 0$ such that $T_0(t_0)(\mathcal{O}_{\sigma_1}(z)) \subset \mathcal{O}_{\frac{\delta}{2}}(A_0^*)$. From the continuity of $\{T_\eta(t) : t \geq 0\}_{\eta\in[0,1]}$ at $\eta = 0$, let $\eta_1 \in (0,\eta_0]$ and $\sigma_2 \in (0,\sigma_1]$ such that $T_\eta(t_0)(\mathcal{O}_{\sigma_2}(z)) \subset \mathcal{O}_\delta(A_0^*)$ for all $\eta \in [0,\eta_1]$.

Finally, from (3.12) we deduce that $l_\eta(T_\eta(t_0)(\mathcal{O}_{\sigma_2}(z))) \subset (1 - \frac{\varepsilon}{2}, 1]$ for all $\eta \in [0,\eta_1]$, so that, for all $w \in \mathcal{O}_{\sigma_2}(z)$ and all $\eta \in [0,\eta_1]$, we have that $1 - \frac{\varepsilon}{2} < l_\eta(T_\eta(t_0)\,w) \leq k_\eta(w) \leq 1$, from which $|k_\eta(w) - k_0(w)| \leq \varepsilon$ for $\eta \in [0,\eta_1]$ and $w \in \mathcal{O}_{\sigma_2}(z)$. Thus

$$\sup_{w\in\mathcal{O}_{\sigma_2}(z)} |k_\eta(w) - k_0(w)| \leq \varepsilon \text{ for } \eta \in [0,\eta_1], \tag{3.13}$$

where $\sigma_2 > 0$ and η_1 that depends only on z and $\varepsilon > 0$.

Now, from cases 1, 2 and 3 we obtain that:

Given a compact subset $K \subset X$, and $\varepsilon > 0$, by (3.6), (3.11) and (3.13), there exist an open subset $U = U(\varepsilon, K) \subset X$ with $K \subset U$, and an index $\eta' = \eta'(\varepsilon, K) > 0$ such that

$$\sup_{w\in U} |k_\eta(w) - k_0(w)| \leq \varepsilon \text{ for all } \eta \in [0,\eta'],$$

and then $\lim\limits_{\eta\to 0^+} \sup\limits_{w\in K} |k_\eta(w) - k_0(w)| = 0$.

Step 3: $h_\eta \overset{\eta\to 0^+}{\to} h_0$ uniformly in compact subsets of X.

Indeed, given $z \in X$ consider now two cases:

Case 1: $\operatorname{dist}(z, \mathcal{A}_0) > 0$.

Given $\alpha > 0$ with $0 < \alpha < \operatorname{dist}(z, \mathcal{A}_0)$, let, by Lemma 3.2, $\alpha' \in (0, \alpha)$ and $\eta_0 \in (0,1]$ such that for all $\eta \in [0, \eta_0]$

$$\gamma_\eta^+ \left(\mathcal{O}_{\alpha'}\left(\mathcal{A}_\eta\right)\right) \subset \mathcal{O}_\alpha\left(\mathcal{A}_\eta\right). \tag{3.14}$$

Choose $t_0 > 0$ such that $T_0(t_0) z \in \mathcal{O}_{\frac{\alpha'}{4}}(\mathcal{A}_0)$ and by continuity of $T_0(t_0) : X \to X$ let $\sigma_1 > 0$ such that $T_0(t_0)(\mathcal{O}_{\sigma_1}(z)) \subset \mathcal{O}_{\frac{\alpha'}{4}}(\mathcal{A}_0)$.

Now, from the continuity of $\{T_\eta(t) : t \geq 0\}_{\eta \in [0,1]}$ at $\eta = 0$, let $\eta_1 \in (0, \eta_0]$ and $\sigma_2 \in (0, \sigma_1]$ such that $T_\eta(t_0)(\mathcal{O}_{\sigma_2}(z)) \subset \mathcal{O}_{\frac{\alpha'}{2}}(\mathcal{A}_0)$ for each $\eta \in [0, \eta_1]$, and, by the lower semicontinuity of $(\mathcal{A}_\eta)_{\eta \in [0,1]}$ at $\eta = 0$, let $\eta_2 \in (0, \eta_1]$ such that $\mathcal{A}_0 \subset \mathcal{O}_{\frac{\alpha'}{2}}(\mathcal{A}_\eta)$ for all $\eta \in [0, \eta_2]$, so that $\mathcal{O}_{\frac{\alpha'}{2}}(\mathcal{A}_0) \subset \mathcal{O}_{\alpha'}(\mathcal{A}_\eta)$ if $\eta \in [0, \eta_2]$. Thus, for all $\eta \in [0, \eta_2]$ we have that $T_\eta(t_0)(\mathcal{O}_{\sigma_2}(z)) \subset \mathcal{O}_{\alpha'}(\mathcal{A}_\eta)$ and from (3.14) we obtain that $\gamma_\eta^+(T_\eta(t_0)(\mathcal{O}_{\sigma_2}(z))) \subset \mathcal{O}_\alpha(\mathcal{A}_\eta)$ for all $\eta \in [0, \eta_2]$. Thus

$$\sup_{t \geq t_0} \operatorname{dist}(T_\eta(t) w, \mathcal{A}_\eta) \leq \alpha \text{ for all } \eta \in [0, \eta_2] \text{ and all } w \in \mathcal{O}_{\sigma_2}(z). \tag{3.15}$$

On the other hand, for all $w \in X$ and all $\eta \in [0,1]$ we have

$$|\operatorname{dist}(w, \mathcal{A}_\eta) - \operatorname{dist}(w, \mathcal{A}_0)| \leq \mathrm{d_H}(\mathcal{A}_\eta, \mathcal{A}_0). \tag{3.16}$$

Then, we can choose $\eta_3 \in (0, \eta_2]$ and $\sigma_3 \in (0, \sigma_2]$ such that $\operatorname{dist}(w, \mathcal{A}_\eta) > \alpha$ for all $\eta \in [0, \eta_3]$ and all $w \in \mathcal{O}_{\sigma_3}(z)$, from which, by (3.15), it follows

$$\sup_{t \geq t_0} \operatorname{dist}(T_\eta(t) w, \mathcal{A}_\eta) \leq \alpha < \operatorname{dist}(w, \mathcal{A}_\eta) \text{ for all } \eta \in [0, \eta_3] \text{ and all } w \in \mathcal{O}_{\sigma_3}(z),$$

so that $h_\eta(w) = \sup_{0 \leq t \leq t_0} \operatorname{dist}(T_\eta(t) w, \mathcal{A}_\eta)$ for each $\eta \in [0, \eta_3]$ and each $w \in \mathcal{O}_{\sigma_3}(z)$.

Note that, for all $w \in X$, all $\eta \in [0,1]$ and all $t \geq 0$ we have that

$$|\operatorname{dist}(T_\eta(t) w, \mathcal{A}_\eta) - \operatorname{dist}(T_0(t) w, \mathcal{A}_0)| \leq \mathrm{d_H}(\mathcal{A}_\eta, \mathcal{A}_0) + d(T_\eta(t) w, T_0(t) w),$$

so that, for all $\eta \in [0, \eta_3]$

$$\sup_{w \in \mathcal{O}_{\sigma_3}(z)} |h_\eta(w) - h_0(w)| \leq \mathrm{d_H}(\mathcal{A}_\eta, \mathcal{A}_0) + \sup_{w \in \mathcal{O}_{\sigma_3}(z)} \sup_{0 \leq t \leq t_0} d(T_\eta(t) w, T_0(t) w),$$

and so, it is easy to see that, given $\varepsilon > 0$ there exist $\sigma \in (0, \sigma_3]$ and $\eta_4 \in (0, \eta_3]$, depending only on z and $\epsilon > 0$, such that

$$\sup_{w \in \mathcal{O}_\sigma(z)} |h_\eta(w) - h_0(w)| \leq \varepsilon \text{ for all } \eta \in [0, \eta_4].$$

Case 2: $\operatorname{dist}(z, \mathcal{A}_0) = 0$, i.e, $z \in \mathcal{A}_0$.

In this case, given $\varepsilon > 0$, by Lemma 3.2, let $\varepsilon' \in \left(0, \frac{\varepsilon}{2}\right)$ and $\eta_0 \in (0,1]$ such that, for all $\eta \in [0, \eta_0]$,

$$\gamma_\eta^+ \left(\mathcal{O}_{\varepsilon'}\left(\mathcal{A}_\eta\right)\right) \subset \mathcal{O}_{\frac{\varepsilon}{2}}\left(\mathcal{A}_\eta\right). \tag{3.17}$$

On the other hand, by the lower semicontinuity of $(\mathcal{A}_\eta)_{\eta\in[0,1]}$ at $\eta = 0$, let $\eta_1 \in (0, \eta_0]$ such that $\mathcal{A}_0 \subset \mathcal{O}_{\frac{\varepsilon'}{2}}(\mathcal{A}_\eta)$ if $\eta \in [0, \eta_1]$, and then, $\mathcal{O}_{\frac{\varepsilon'}{2}}(\mathcal{A}_0) \subset \mathcal{O}_{\varepsilon'}(\mathcal{A}_\eta)$ if $\eta \in [0, \eta_1]$, and thus (3.17) implies that

$$\gamma_\eta^+ \left(\mathcal{O}_{\frac{\varepsilon'}{2}}(\mathcal{A}_0) \right) \subset \mathcal{O}_{\frac{\varepsilon}{2}}(\mathcal{A}_\eta) \text{ if } \eta \in [0, \eta_1],$$

from which $h_\eta(w) = \sup_{t\geq 0} \operatorname{dist}(T_\eta(t)w, \mathcal{A}_\eta) \leq \frac{\varepsilon}{2}$ for all $\eta \in [0, \eta_1]$ and $w \in \mathcal{O}_{\frac{\varepsilon'}{2}}(\mathcal{A}_0)$, so that we conclude

$$\sup_{w\in\mathcal{O}_{\frac{\varepsilon'}{2}}(\mathcal{A}_0)} |h_\eta(w) - h_0(w)| \leq \varepsilon \text{ for all } \eta \in [0, \eta_1].$$

In these conditions, given $\varepsilon > 0$, each $z \in X$ possesses a neighbourhood $\mathcal{O}_\sigma(z)$, with $\sigma = \sigma(\varepsilon, z) > 0$, and there exists an index $\eta' = \eta'(\varepsilon, z) > 0$ such that

$$\sup_{w\in\mathcal{O}_\sigma(z)} |h_\eta(w) - h_0(w)| \leq \varepsilon \text{ if } \eta \in [0, \eta'],$$

so that we conclude the convergence of $h_\eta \overset{\eta\to 0^+}{\to} h_0$ uniformly in compact sets of X by a similar argument to the one in Step 2. This completes the proof. □

Remark 3.5. *Let $\{T_\eta(t) : t \geq 0\}_{\eta\in[0,1]}$ be a family of semigroups in a metric space X satisfying hypotheses of Theorem 2.22 with isolated invariant sets $\mathbf{\Xi}_\eta := \{\Xi_{1,\eta}, \cdots, \Xi_{n,\eta}\}$ reordered in such a way that Ξ_j is a local attractor for the restriction of $\{T_\eta(t) : t \geq 0\}$ to $(\Xi_\eta)^*_{j-1,j-2}$. For suitably small η, $\{T_\eta(t) : t \geq 0\}$ is a gradient-like semigroup associated to $\mathbf{\Xi}_\eta$, and such that $\mathbf{\Xi}_\eta$ is a Morse decomposition with associated local attractors*

$A_{0,\eta} := \varnothing$ and for each $j = 1, 2, \cdots, n$

$$A_{j,\eta} := \bigcup_{i=1}^{j} W_\eta^u(\Xi_{i,\eta}).$$

Then, the repellers are given by:

$A^*_{n,\eta} := \varnothing$ and for each $j = 0, 1, \cdots, n-1$

$$A^*_{j,\eta} := \bigcup_{i=j+1}^{n} W_\eta^s(\Xi_{i,\eta}),$$

where, for each $\eta \in [0,1]$ and all $i = 1, 2, \cdots, n$

$$W_\eta^s(\Xi_{i,\eta}) := \left\{ z \in \mathcal{A}_\eta : T_\eta(t) z \overset{t\to\infty}{\to} \Xi_{i,\eta} \right\}$$

and $W^s_{loc,\eta}(\Xi_{i,\eta})$ or $W^u_{loc,\eta}(\Xi_{i,\eta})$ (in the context of generalized gradient-like semigroups) is the intersection of $W_\eta^s(\Xi_{i,\eta})$ or $W_\eta^u(\Xi_{i,\eta})$ with a neighborhood of $\Xi_{i,\eta}$.

Thus, we must look for sufficient conditions to obtain continuity of stable (restricted to the attractors) and unstable manifolds in order to obtain the continuity of Lyapunov functions.

This last remark leads us to the following result:

Corollary 3.6. *Let $\{T_\eta(t) : t \geq 0\}_{\eta\in[0,1]}$ be a family of semigroups in a metric space (X, d) satisfying the hypotheses of Theorem 2.22 with isolated invariant sets $\boldsymbol{\Xi}_\eta := \{\Xi_{1,\eta}, \cdots, \Xi_{n,\eta}\}$.*

If the stable $\left(W^s_\eta(\Xi_{j,\eta})\right)_{\eta\in[0,1]}$ and unstable $\left(W^u_\eta(\Xi_{j,\eta})\right)_{\eta\in[0,1]}$ manifolds are continuous at $\eta = 0$, for all $j = 1, 2, \cdots, n$, then the Lyapunov functions associated to $\{T_\eta(t) : t \geq 0\}$, given with the aid of Proposition 3.4, for η small enough behave continuously at $\eta = 0$.

4. Energy level decomposition of a generalized gradient-like semigroup

We now give a dynamical description of a generalized gradient-like semigroup by reordering and regrouping the corresponding isolated invariant subsets to obtain a *totally ordered* family of isolated invariant sets that we will refer to as *energy levels*. This new family of isolated invariant sets is a Morse decomposition of $\mathcal{A}$ with fewer invariant sets but in such a way that it still gives us a Lyapunov function that is constant only in the solutions lying in the original isolated invariant sets. In a certain sense, this decomposition is the coarsest decomposition which still gives us a Lyapunov function which is constant only in the solutions lying in the original isolated invariant sets.

Assume that $\{T(t) : t \geq 0\}$ is a generalized gradient-like semigroup with respect to the disjoint family of isolated invariant sets $\boldsymbol{\Xi} = \{\Xi_1, \Xi_2, \cdots, \Xi_n\}$.

(a) Given Ξ_{l_1} and $\Xi_{l_2} \in \boldsymbol{\Xi}$, we say that Ξ_{l_1} precedes Ξ_{l_2} (we write $\Xi_{l_1} \prec \Xi_{l_2}$), if there exists a global solution $\xi : \mathbb{R} \to X$ of $\{T(t) : t \geq 0\}$ such that $\xi(\mathbb{R}) \nsubseteq \Xi_{l_1} \cup \Xi_{l_2}$ and

$$\lim_{t\to-\infty} d(\xi(t), \Xi_{l_2}) = 0 \text{ and } \lim_{t\to\infty} d(\xi(t), \Xi_{l_1}) = 0.$$

(b) Let us consider

$$\mathcal{M}_1 := \{\Xi_\ell \in \boldsymbol{\Xi} : \text{ there is no element } \Xi \in \boldsymbol{\Xi} \text{ that preceeds } \Xi_\ell\}$$

and, for any integer $k \geq 2$

$$\mathcal{M}_k := \{\Xi_\ell \in \boldsymbol{\Xi} : \text{ if } \Xi \in \boldsymbol{\Xi} \text{ and } \Xi \prec \Xi_\ell \text{ then } \Xi \in \mathcal{M}_{k-1}\}.$$

Note that, by definition, $\mathcal{M}_k \subset \mathcal{M}_{k+1}$.

(c) We now define the sets

$$\mathcal{N}_1 := \bigcup_{\Xi\in\mathcal{M}_1} \Xi, \text{ and } \mathcal{N}_k := \bigcup_{\Xi\in\mathcal{M}_k\setminus\mathcal{M}_{k-1}} \Xi, \text{ for all } k \geq 2.$$

Since $\boldsymbol{\Xi}$ is finite, there exists a positive integer q such that $\mathcal{M}_k = \mathcal{M}_q$ for each $k > q$, so that, $\mathcal{N}_k = \varnothing$ for all $k > q$. Thus, let $\mathcal{N}_1, \mathcal{N}_2, \cdots, \mathcal{N}_p$, the level sets with $p := \min\{q \in \mathbb{N} : \mathcal{M}_k = \mathcal{M}_q \text{ for each } k > q\}$.

We have the following first result related to this family of sets:

Lemma 4.1. *(see [1]) Let $\{T(t) : t \geq 0\}$ be a semigroup with global attractor $\mathcal{A}$. Assume that $\{T(t) : t \geq 0\}$ is a generalized gradient-like semigroup with respect to the disjoint family of isolated invariant sets $\mathbf{\Xi} = \{\Xi_1, \Xi_2, \cdots, \Xi_n\}$. Then each element of $\mathbf{\Xi}$ is contained in $\mathcal{N}_k$, for some $k \leq p$.*

The following result will show that $\mathcal{N} = (\mathcal{N}_1, \mathcal{N}_2, \cdots, \mathcal{N}_p)$ is a Morse decomposition for $\mathcal{A}$.

Theorem 4.2. *Let $\{T(t) : t \geq 0\}$ be a semigroup with global attractor $\mathcal{A}$. If $\{T(t) : t \geq 0\}$ is a generalized gradient-like semigroup with respect to $\mathbf{\Xi} = \{\Xi_1, \Xi_2, \cdots, \Xi_n\}$, then $(\mathcal{N}_1, \mathcal{N}_2, \cdots, \mathcal{N}_p)$ is a Morse decomposition for $\mathcal{A}$.*

Proof. Clearly $\{T(t) : t \geq 0\}$ is a generalized gradient-like semigroup with respect to $\mathcal{N}$. The proof of the result now follows from Theorem 2.16 (see [1]). □

In order to see that the continuity of local unstable manifolds is not sufficient to obtain the continuity of Lyapunov functions one may consider the example in the following picture

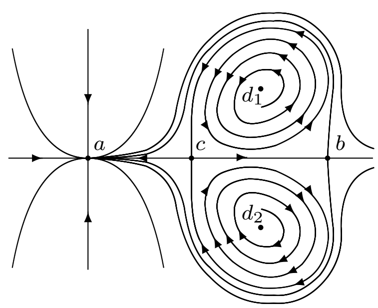

Figure 01

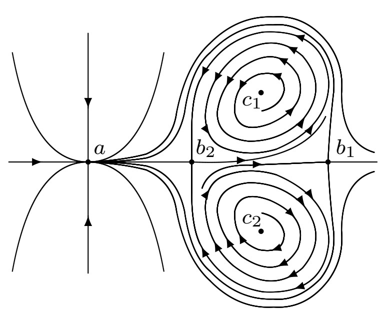

Figure 02

Note that, in both cases the semigroup associated are gradient-like. Also, the semigroup associated to Figure 01 has energy levels $\mathcal{N}_1 = \{a\}$, $\mathcal{N}_2 = \{b\}$, $\mathcal{N}_3 = \{c\}$, $\mathcal{N}_4 = \{d_1, d_2\}$, while the semigroup associated to Figure 02 has energy levels $\mathcal{N}_1 = \{a\}$, $\mathcal{N}_2 = \{b_1, b_2\}$, $\mathcal{N}_3 = \{c_1, c_2\}$. This clearly shows that, even if all equilibria are hyperbolic, if the connections between them are not stable under perturbations, the level sets may be discontinuous.

Remark 4.3. *All the concepts and results in the previous section can be written in the particular case in which we have a finite set of equilibria. Indeed, let* $\{T(t) : t \geq 0\}$ *be a gradient-like semigroup in* X *with global attractor* $\mathcal{A}$ *with equilibrium points* $\mathcal{E} = \{\zeta_1, \cdots, \zeta_n\}$*. Then, there exists an energy level decomposition in* $\mathcal{A}$ *made of equilibrium points.*

5. Energy levels for a generalized gradient-like semigroup under perturbation

Again, for each $\eta \in [0,1]$, let $\{T_\eta(t) : t \geq 0\}$ a semigroup on a metric space X, with global attractor $\mathcal{A}_\eta$, and a finite family of isolated bounded sets $\mathbf{\Xi}_\eta = \{\Xi_{1,\eta}, \Xi_{2,\eta}, \cdots, \Xi_{p,\eta}\}$, such that each $\{T_\eta(t) : t \geq 0\}$ is a generalized gradient-like semigroup with respect to $\mathbf{\Xi}_\eta$. We suppose

$$\sup_{1 \leq i \leq p} \mathrm{d_H}\left(\Xi_{i,\eta}, \Xi_{i,0}\right) \overset{\eta \to 0+}{\to} 0.$$

Under these conditions, we give sufficient conditions so that the energy levels are continuous under perturbation.

Lemma 5.1. *For each* $\eta \in [0,1]$*, let* $\{T_\eta(t) : t \geq 0\}$ *be a semigroup on a metric space* X*, with global attractor* $\mathcal{A}_\eta$ *and a finite family of isolated bounded sets* $\mathbf{\Xi}_\eta = \{\Xi_{1,\eta}, \Xi_{2,\eta}, \cdots, \Xi_{n,\eta}\}$*, such that each* $\{T_\eta(t) : t \geq 0\}$ *is a generalized gradient-like semigroup with respect to* $\mathbf{\Xi}_\eta$*, and let* $\mathbf{N}_\eta = (\mathcal{N}_{1,\eta}, ..., \mathcal{N}_{p(\eta),\eta})$ *be the corresponding Morse decomposition formed by the energy levels. Assume the hypotheses of Theorem 2.22. Let* $\Xi_0 \in \mathcal{N}_{1,0}$ *and* $(\Xi_\eta)_{\eta \in (0,1]}$*, with* $\Xi_\eta \in \mathbf{\Xi}_\eta$ *for each* $\eta \in (0,1]$*, the unique family such that* $\mathrm{d_H}(\Xi_\eta, \Xi_0) \overset{\eta \to 0+}{\to} 0$.

Then there exist $\delta > 0$ *and* $\eta_1 \in (0,1]$ *such that, for any* $\eta \in (0, \eta_1]$*, if* $z \in X$ *is such that* $\mathrm{dist}(z, \Xi_\eta) < \delta$ *then* $\mathrm{dist}(T_\eta(t) z, \Xi_\eta) \overset{t \to \infty}{\to} 0$*. Moreover, for* $i \geq 2$ *if* $\Xi_0 \in \mathcal{N}_{i,0}$ *and* $(\Xi_\eta)_{\eta \in (0,1]}$*, with* $\Xi_\eta \in \mathbf{\Xi}_\eta$ *for each* $\eta \in (0,1]$*, is the unique family such that* $\mathrm{d_H}(\Xi_\eta, \Xi_0) \overset{\eta \to 0+}{\to} 0$*, then there exist* $\delta > 0$ *and* $\eta_i \in (0,1]$ *such that for any* $\eta \in (0, \eta_i]$*, if* $z \in X$ *satisfies* $\mathrm{dist}(z, \Xi_\eta) < \delta$*, then, either* $\mathrm{dist}(T_\eta(t) z, \Xi_\eta) \overset{t \to \infty}{\to} 0$*, or* $\mathrm{dist}\left(T_\eta(t) z, \mathcal{M}_{(i-1),\eta}\right) \overset{t \to \infty}{\to}$*, where* $\mathcal{M}_{(i-1),\eta} := \bigcup_{j=1}^{i-1} \mathcal{N}_{j,\eta}$*, in particular* $\mathrm{dist}(T_\eta(t) z, \mathcal{M}_{i,\eta}) \xrightarrow{t \to 0} 0$.

Proof. For the case $\Xi_0 \in \mathcal{N}_{1,0}$, suppose not. Then, there exist $\eta_k \to 0^+$ in $(0,1]$, $(z_k)_{k \in \mathbb{N}}$ in X and $(\Xi_{\eta_k})_{k \in \mathbb{N}}$ subfamily of $(\Xi_\eta)_{\eta \in (0,1]}$ with $d(z_k, \Xi_{\eta_k}) < \frac{1}{k}$ such that $T_{\eta_k}(t) z_k$ does not converge to Ξ_{η_k} when $t \to \infty$, for all k. Fix $\delta_0 > 0$ such that $\mathcal{O}_{\delta_0}(\Xi_{i,\eta}) \cap \mathcal{O}_{\delta_0}(\Xi_{j,\eta}) = \varnothing$ for $i \neq j$ and η small enough. Then, as each $\{T_{\eta_k}(t) : t \geq 0\}$ is a generalized gradient-like semigroup with respect to $\mathbf{\Xi}_{\eta_k}$, so that, for each k, $\mathrm{dist}\left(T_{\eta_k}(t) z_k, \Xi_{\eta_k}^{(k)}\right) \overset{t \to \infty}{\to} 0$ for some $\Xi_{\eta_k}^{(k)} \in \mathbf{\Xi}_{\eta_k} \setminus \{\Xi_{\eta_k}\}$ and so for k big enough, we can find $\tau_k > 0$ such that

$$\mathrm{dist}(T_{\eta_k}(t) z_k, \Xi_{\eta_k}) < \delta_0 \text{ for } t \in [0, \tau_k) \text{ and} \tag{5.1}$$

$$\mathrm{dist}(T_{\eta_k}(\tau_k) z_k, \Xi_{\eta_k}) = \delta_0. \tag{5.2}$$

By the uniform convergence $T_\eta \overset{\eta\to 0^+}{\to} T_0$ on compacts of $[0,\infty)\times X$, from $\mathrm{d_H}\left(\Xi_{\eta_k},\Xi_0\right) \overset{k\to\infty}{\to} 0$ and by (5.2) we have that $\tau_k \overset{k\to\infty}{\to} \infty$. Thus, consider, for each k big enough, the map $\xi_k : [-\tau_k,\infty) \to X$ given by $\xi_k\left(t\right) := T_{\eta_k}\left(t+\tau_k\right) z_k$ $t \in [-\tau_k,\infty)$. By the collective compactness and from (5.1) there exists a global bounded solution $\xi_0 : \mathbb{R} \to X$ for $\{T_0\left(t\right) : t \geq 0\}$ such that $\lim_{k\to\infty} \xi_k\left(t\right) = \xi_0\left(t\right)$ for all $t \in \mathbb{R}$ and $\lim_{t\to-\infty} \mathrm{dist}\left(\xi_0\left(t\right),\Xi_0\right) = 0$. But, as Ξ_0 in $\mathcal{N}_{1,0}$ with $\{T_0\left(t\right) : t \geq 0\}$ generalized gradient-like, we have $\xi_0\left(t\right) \in \Xi_0$ for all $t \in \mathbb{R}$, which contradicts $d\left(\xi_0\left(0\right),\Xi_0\right) = \delta_0$, which comes from (5.2) as $k \to \infty$.

For $i = 2$, we also argue by contradiction. Then we obtain $\eta_k \to 0^+$ in $(0,1]$, $(z_k)_{k\in\mathbb{N}}$ in X and $\left(\Xi_{\eta_k}\right)_{k\in\mathbb{N}}$ subfamily of $\left(\Xi_\eta\right)_{\eta\in(0,1]}$ with $\mathrm{dist}\left(z_k,\Xi_{\eta_k}\right) < \frac{1}{k}$ such that $T_{\eta_k}\left(t\right) z_k$ does not converge to Ξ_{η_k} when $t \to \infty$ and $T_{\eta_k}\left(t\right) z_k$ does not converge to $\mathcal{N}_{1,\eta_k}$ when $t \to \infty$. Now, let $\delta > 0$ such that the conclusion of the previous case is satisfied in $\mathcal{O}_\delta\left(\mathcal{N}_{1,\eta_k}\right)$ for all k big enough and with $\mathcal{O}_\delta\left(\Xi_{i,\eta}\right) \cap \mathcal{O}_\delta\left(\Xi_{j,\eta}\right) = \varnothing$ for $i \neq j$ and η small enough.

Thus, for all $t \geq 0$ and k we have

$$\mathrm{dist}\left(T_{\eta_k}\left(t\right) z_k,\mathcal{N}_{1,\eta_k}\right) \geq \delta. \tag{5.3}$$

On the other hand, as each $\{T_{\eta_k}\left(t\right) : t \geq 0\}$ is a generalized gradient-like semigroup, for each k we have that $\mathrm{dist}\left(T_{\eta_k}\left(t\right) z,\Xi^{(k)}_{\eta_k}\right) \overset{t\to\infty}{\to} 0$ for some $\Xi^{(k)}_{\eta_k} \in \mathbf{\Xi}_{\eta_k} \setminus \{\Xi_{\eta_k}\}$ and, consequently, for each k large enough, there exists $\tau_k > 0$ satisfying

$$\mathrm{dist}\left(T_{\eta_k}\left(t\right) z_k,\Xi_{\eta_k}\right) < \delta \text{ for } t \in [0,\tau_k) \text{ and} \tag{5.4}$$

$$\mathrm{dist}\left(T_{\eta_k}\left(\tau_k\right) z_k,\Xi_{\eta_k}\right) = \delta. \tag{5.5}$$

Again, by (5.5), it holds $\tau_k \overset{k\to\infty}{\to} \infty$ and then, if we define $\xi_k : [-\tau_k,\infty) \to X$ given by $\xi_k\left(t\right) := T_{\eta_k}\left(t+\tau_k\right) z_k$ $t \in [-\tau_k,\infty)$, from the collective compactness and (5.4), we obtain the existence of a global bounded solution for $\{T_0\left(t\right) : t \geq 0\}$, $\xi_0 : \mathbb{R} \to X$, such that $\lim_{k\to\infty} \xi_k\left(t\right) = \xi_0\left(t\right)$ for all $t \in \mathbb{R}$ with $\lim_{t\to-\infty} d\left(\xi_0\left(t\right),\Xi_0\right) = 0$. Since $\{T_0\left(t\right) : t \geq 0\}$ is a generalized gradient-like semigroup with respect to $\mathbf{\Xi}_0$, there exists $\Xi_{l_1} \in \mathbf{\Xi}_0$ with $\lim_{t\to\infty} \mathrm{dist}\left(\xi_0\left(t\right),\Xi_{l_1}\right) = 0$. As $\Xi_0 \in \mathcal{N}_{2,0}$, it holds that $\Xi_{l_1} \in \mathcal{N}_{1,0}$, from where, for $\tau > 0$ with $\mathrm{dist}\left(\xi_0\left(\tau\right),\Xi_{l_1}\right) < \frac{\delta}{2}$, we deduce, for k big enough, $\mathrm{dist}\left(T_{\eta_k}\left(\tau+\tau_k\right) z_k,\mathcal{N}_{1,\eta_k}\right) < \frac{\delta}{2}$, which contradicts (5.3). A similar argument for the remaining cases finishes the proof. $\square$

Theorem 5.2. *Suppose the hypotheses of the previous lemma. Let $\mathcal{N}_{1,\eta},\cdots,\mathcal{N}_{p(\eta),\eta}$ the energy levels associated to the family $\mathbf{\Xi}_\eta = \{\Xi_{1,\eta},\cdots,\Xi_{n,\eta}\}$ for $\eta \in (0,1]$, and suppose that,*

- **(H)** *if $(\eta_k)_{k\in\mathbb{N}}$ is a sequence in $(0,1]$ with $\eta_k \overset{k\to\infty}{\to} 0^+$ and $\left(\Xi_{\eta_k}\right)_{k\in\mathbb{N}}$ satisfy that, for some $i \in \bigcap_{k\in\mathbb{N}} \{1,2,\cdots,p(\eta_k)\}$, $\Xi_{\eta_k} \in \mathcal{N}_{i,\eta_k}$ and $\mathrm{d_H}\left(\Xi_{\eta_k},\Xi_0\right) \overset{k\to\infty}{\to} 0$ then $\Xi_0 \in \mathcal{N}_{i,0}$.*

Then, if p denotes the number of energy levels for $\{T_0(t) : t \geq 0\}$, written as $\mathcal{N}_{1,0},\cdots,\mathcal{N}_{p,0}$, there exists $\eta^* \in (0,1]$ such that for all $\eta \in (0,\eta^*]$ the semigroup $\{T_\eta\left(t\right) : t \geq 0\}$ possesses *also p energy levels, $\mathcal{N}_{1,\eta},\cdots,\mathcal{N}_{p,\eta}$ (i.e., $p(\eta)=p$ for all $\eta\in(0,\eta^*]$), and*

$$\mathrm{d_H}\left(\mathcal{N}_{i,\eta},\mathcal{N}_{i,0}\right)\overset{\eta\to 0^+}{\to} 0 \text{ for all } i=1,2,\cdots,p.$$

Proof. Let us write the energy levels for the limit case.

For $i=1,2,\cdots,p$, $\mathcal{N}_{i,0}=\left\{\Xi^{(i)}_{l_1,0},\cdots,\Xi^{(i)}_{l_{k(i)},0}\right\}$.

If we define, for each $\eta\in(0,1]$ and $i=1,2,\cdots,p$ the sets $\mathcal{H}_{i,\eta}:=\left\{\Xi^{(i)}_{l_1,\eta},\cdots,\Xi^{(i)}_{l_{k(i)},\eta}\right\}$ and $\mathcal{H}'_{i,\eta}:=\bigcup_{j=1}^{k(i)}\Xi^{(i)}_{l_j,\eta}$, then, Theorem 2.22 implies $\mathrm{d_H}\left(\mathcal{H}'_{i,\eta},\mathcal{N}_{i,0}\right)\overset{\eta\to 0^+}{\to}0$, for all $i=1,2,\cdots,p$.

The sets $\mathcal{H}_{i,\eta}$'s are the natural candidates to be the energy levels for $T_\eta(\cdot)$. Indeed, let us prove that it holds that $\mathcal{H}_{i,\eta}=\mathcal{N}_{i,\eta}$, for $i=1,2,\cdots,p$ and η small enough, i.e., $\mathcal{H}_{1,\eta},\mathcal{H}_{2,\eta},\cdots,\mathcal{H}_{p,\eta}$ are the energy levels of $\{T_\eta(t):t\geq 0\}$ for η small enough.

For $i=1$ let $\left(\Xi^{(1)}_{l_1,\eta}\right)_{\eta\in(0,1]}$ be the family within the set $\mathcal{H}_{1,\eta}$. Then, there exists $\theta_1\in(0,1]$ such that $\Xi^{(1)}_{l_1,\eta}\in\mathcal{N}_{1,\eta}$ for all $\eta\in(0,\theta_1]$. Indeed, if not, we can find a sequence $\eta_k\to 0^+$ and global solutions $\xi_k:\mathbb{R}\to X$ for $\{T_{\eta_k}(t):t\geq 0\}$ such that $\lim_{t\to-\infty}\operatorname{dist}\left(\xi_k(t),\Xi^{(1)}_{l_j,\eta_k}\right)=0$ and $\lim_{t\to\infty}\operatorname{dist}\left(\xi_k(t),\Xi_{\eta_k}\right)=0$, for some isolated invariant set $\Xi_{\eta_k}\in\mathbf{\Xi}_{\eta_k}$, but with $\Xi_{\eta_k}\neq\Xi^{(1)}_{l_1,\eta_k}$ for all k.

Choose now, for each k big enough, τ_k such that $\operatorname{dist}\left(\xi_k(t),\Xi^{(1)}_{l_j,\eta_k}\right)<\delta_0$ for all $t<\tau_k$ and

$$\operatorname{dist}\left(\xi_k(\tau_k),\Xi^{(1)}_{l_1,\eta_k}\right)=\delta_0, \tag{5.6}$$

where $\delta_0>0$ satisfies $\mathcal{O}_{\delta_0}(\Xi_{i,\eta})\cap\mathcal{O}_{\delta_0}(\Xi_{j,\eta})=\varnothing$ for $i\neq j$ and η small enough.

If we define for k big enough, $\zeta_k:\mathbb{R}\to X$ by $\zeta_k(t):=\xi_k(t+\tau_k)$ $t\in\mathbb{R}$, we get $\zeta_0:\mathbb{R}\to X$ a global solution of $\{T_0(t):t\geq 0\}$ and a subsequence of $(\zeta_k)_{k\in\mathbb{N}}$, written the same, satisfying $\lim_{k\to\infty}\operatorname{dist}(\zeta_k(t),\zeta_0(t))=0$ for all $t\in\mathbb{R}$, with $\lim_{t\to-\infty}\operatorname{dist}\left(\zeta_0(t),\Xi^{(1)}_{l_1,0}\right)=0$. Since $\Xi^{(1)}_{l_1,0}\in\mathcal{N}_{1,0}$, from the definition of $\mathcal{N}_{1,0}$ it follows that $\zeta_0(t)\in\Xi^{(1)}_{l_1,0}$ for all $t\in\mathbb{R}$. But this fact contradicts that $\operatorname{dist}\left(\zeta_0(0),\Xi^{(1)}_{l_1,0}\right)=\delta_0$, which comes from (5.6) as $k\to\infty$. The same argument for $j=2,\cdots,k(1)$ leads to $\eta_1\in(0,1]$ such that $\Xi^{(1)}_{l_j,\eta}\in\mathcal{N}_{1,\eta}$ for $j=1,\cdots,k(1)$ y $\eta\in(0,\eta_1]$, that is, $\mathcal{H}_{1,\eta}\subset\mathcal{N}_{1,\eta}$ for all $\eta\in(0,\eta_1]$.

On the other hand, there exists $\eta_1'\in(0,\eta_1]$ such that $\mathcal{N}_{1,\eta}\subset\mathcal{H}_{1,\eta}$ when $\eta\in(0,\eta_1']$. If not, there exist a sequence $\eta_k\to 0^+$ and, for each k, an isolated invariant set $\Xi_{\eta_k}\in\mathcal{N}_{1,\eta_k}\backslash\mathcal{H}_{1,\eta_k}$ such that the sequence $(\Xi_{\eta_k})_{k\in\mathbb{N}}$ satisfies $\mathrm{d_H}(\Xi_{\eta_k},\Xi_0)\overset{k\to\infty}{\to}0$. However, from (**H**), we have $\Xi_0\in\mathcal{N}_{1,0}$, which contradicts that $\Xi_{\eta_k}\notin\mathcal{H}_{1,\eta_k}$ for any k. Thus, we conclude that $\mathcal{H}_{1,\eta}=\mathcal{N}_{1,\eta}$ for $\eta\in(0,\eta_1']$.

We now show that there exists $\eta_2\in(0,\eta_1']$ such that $\mathcal{H}_{2,\eta}\subset\mathcal{M}_{2,\eta}$ if $\eta\in(0,\eta_2]$. Note that, if this claim holds, from the proof of the above case we have that $\mathcal{H}_{2,\eta}\subset\mathcal{M}_{2,\eta}\backslash\mathcal{N}_{1,\eta}=\mathcal{N}_{2,\eta}$, once $\mathcal{H}_{2,\eta}$ is disjoint of $\mathcal{H}_{1,\eta}=\mathcal{N}_{1,\eta}$ for all $\eta\in(0,\eta_2]$.

To get the existence of η_2, take the family $\left(\Xi^{(2)}_{l_1,\eta}\right)_{\eta\in(0,\eta_1]}$ of the elements in $\mathcal{H}_{2,\eta}$'s. Then there exists $\theta_2 \in (0,\eta_1]$ such that $\Xi^{(2)}_{l_1,\eta} \in \mathcal{M}_{2,\eta}$ for all $\eta \in (0,\theta_2]$. If not, by the same argument above, we get a subsequence $\eta_k \to 0^+$ and corresponding global solutions $\xi_k : \mathbb{R} \to X$ for $\{T_{\eta_k}(t) : t \geq 0\}$ such that $\lim\limits_{t\to-\infty} \operatorname{dist}\left(\xi_k(t), \Xi^{(2)}_{l_1,\eta_k}\right) = 0$ and $\lim\limits_{t\to\infty} d(\xi_k(t), \Xi_{\eta_k}) = 0$, for some isolated invariant sets $\Xi_{\eta_k} \in \boldsymbol{\Xi}_{\eta_k}$ with $\Xi_{\eta_k} \notin \mathcal{M}_{1,\eta_k} = \mathcal{N}_{1,\eta_k}$ and $\Xi_{\eta_k} \neq \Xi^{(2)}_{l_1,\eta_k}$ for all k. As above and for the same δ_0 let, for each k, $\tau_k \in \mathbb{R}$ such that $\operatorname{dist}\left(\xi_k(t), \Xi^{(2)}_{l_1,\eta_k}\right) < \delta_0$ for all $t < \tau_k$ and $\operatorname{dist}\left(\xi_k(\tau_k), \Xi^{(2)}_{l_1,\eta_k}\right) = \delta_0$.

From lemma 5.1, let $\delta > 0$ and $\bar{\eta}_1 \in (0,\eta_1']$ such that the asymptotic stability of the elements in $\mathcal{N}_{1,\eta}$ are satisfied in $\mathcal{O}_\delta(\mathcal{N}_{1,\eta})$ if $\eta \in [0,\bar{\eta}_1]$. Then,

$$\operatorname{dist}(\xi_k(t), \mathcal{N}_{1,\eta_k}) \geq \delta, \text{ for all } t \in \mathbb{R} \text{ and all } k \in \mathbb{N}. \tag{5.7}$$

If we define the solutions of $\zeta_k : \mathbb{R} \to X$ by $\zeta_k(t) = \xi_k(t+\tau_k)$ $t \in \mathbb{R}$, we get again a global solution $\zeta_0 : \mathbb{R} \to X$ de $\{T_0(t) : t \geq 0\}$ such that $\lim\limits_{k\to\infty} \zeta_k(t) = \zeta_0(t)$ for all $t \in \mathbb{R}$ with $\lim\limits_{t\to-\infty} \operatorname{dist}\left(\zeta_0(t), \Xi^{(2)}_{l_1,0}\right) = 0$. As $\{T_0(t) : t \geq 0\}$ is a generalized gradient-like semigroup, there exists $\Xi_0 \in \boldsymbol{\Xi}_0$ such that $\lim\limits_{t\to\infty} \operatorname{dist}(\zeta_0(t), \Xi_0) = 0$ and since $\Xi^{(2)}_{l_1,0} \in \mathcal{N}_{2,0}$ we get $\Xi_0 \in \mathcal{N}_{1,0}$. Thus, let $\tau > 0$ such that $\operatorname{dist}(\zeta_0(\tau), \Xi_0) < \frac{\delta}{2}$, from which it follows the existence of $k_0 \in \mathbb{N}$ such that $\operatorname{dist}(\xi_k(\tau+\tau_k), \mathcal{N}_{1,\eta_k}) < \delta$ for all $k \geq k_0$, which contradicts (5.7).

The same argument can be used for all $j = 2, \cdots, k(2)$ and so we obtain $\eta_2 \in (0,1]$ such that $\mathcal{H}_{2,\eta} \subset \mathcal{M}_{2,\eta}$ if $\eta \in (0,\eta_2]$.

Again, we get $\eta_2' \in (0,\eta_2]$ such that $\mathcal{N}_{2,\eta} \subset \mathcal{H}_{2,\eta}$ if $\eta \in (0,\eta_2']$, from which we conclude that $\mathcal{H}_{2,\eta} = \mathcal{N}_{2,\eta}$ for all $\eta \in (0,\eta_2']$.

Finally, repeating the reasoning for $i = 3, \cdots, p$ and recalling that $\boldsymbol{\Xi}_\eta = \mathcal{H}_{1,\eta} \cup \cdots \cup \mathcal{H}_{p,\eta}$ for each η, the proof is completed. $\square$

In the following theorem we state a sufficient condition for the hypotheses in the previous result. In particular, we prove that the stability of connecting orbits under perturbation gives the desired result on the continuity of the energy level sets.

Consider $\{T_\eta(t) : t \geq 0\}_{\eta\in[0,1]}$ a family of generalized gradient-like semigroups with respect to $\boldsymbol{\Xi}_\eta = \{\Xi_{1,\eta}, \cdots, \Xi_{n,\eta}\}$ for each $\eta \in [0,1]$. Suppose that:

(HG) For each $\Xi_{l_1,0}, \Xi_{l_2,0} \in \boldsymbol{\Xi}_0$ such that $\Xi_{l_1,0} \prec \Xi_{l_2,0}$, if $\Xi_{l_1,\eta}, \Xi_{l_2,\eta}$ are in $\boldsymbol{\Xi}_\eta$ for $\eta \in (0,1]$ and satisfy $\mathrm{d_H}(\Xi_{l_1,\eta}, \Xi_{l_1,0}) \overset{\eta\to0^+}{\longrightarrow} 0$ and $\mathrm{d_H}(\Xi_{l_2,\eta}, \Xi_{l_2,0}) \overset{\eta\to0^+}{\longrightarrow} 0$, then $\Xi_{l_1,\eta} \prec \Xi_{l_2,\eta}$ for all η small enough.

Theorem 5.3. *Suppose hypotheses in Theorem 2.22, and that (HG) is satisfied for $\{T_\eta(t) : t \geq 0\}_{\eta\in[0,1]}$.*

Then, if $(\eta_k)_{k\in\mathbb{N}}$ *is a sequence in* $(0,1]$ *with* $\eta_k \stackrel{k\to\infty}{\to} 0^+$ *and for some* $i \in \bigcap_{k\in\mathbb{N}}\{1,2,\cdots,p(\eta_k)\}$ $(\Xi_{\eta_k})_{k\in\mathbb{N}}$ *is a sequence with* $\Xi_{\eta_k}\in\mathcal{N}_{i,\eta_k}$ *for all* k *and* $\mathrm{d_H}(\Xi_{\eta_k},\Xi_0)\stackrel{\eta\to 0^+}{\to} 0$ *for some* $\Xi_0 \in \boldsymbol{\Xi}_0$, *then* $\Xi_0 \in \mathcal{N}_{i,0}$.

Proof. Indeed, if for $i=1$ and Ξ_0 does not belong to $\mathcal{N}_{1,0}$ there exits $\Xi_{l_1}\in\boldsymbol{\Xi}_0$ with $\Xi_{l_1}\prec\Xi_0$ but with $\Xi_{l_1}\neq\Xi_0$. Then, let $\left(\Xi_{l_1,\eta_k}\right)_{k\in\mathbb{N}}$ the sequence with $\Xi_{l_1,\eta_k}\in\boldsymbol{\Xi}_{\eta_k}$, for all k, such that $\mathrm{d_H}\left(\Xi_{l_1,\eta},\Xi_{l_1}\right)\stackrel{\eta\to 0^+}{\to} 0$. By (HG) we have $\Xi_{l_1,\eta_k}\prec\Xi_{\eta_k}$ for all k big enough, which contradicts that $\Xi_{\eta_k}\in\mathcal{N}_{1,\eta_k}$. Thus the result is true for $i=1$ and from it and the first part of the proof in Theorem 5.2, we get $\eta_1\in(0,1]$ such that $\mathcal{H}_{1,\eta}$, are $\mathcal{N}_{1,\eta}$ for $\eta\in(0,\eta_1]$.

For $i=2$, if Ξ_0 does not belong to $\mathcal{N}_{2,0}=\mathcal{M}_{2,0}\backslash\mathcal{N}_{1,0}$ we have, on the one hand, that Ξ_0 is not in $\mathcal{N}_{1,0}$, since if $\Xi_0\in\mathcal{N}_{1,0}$, as we have seen above $\mathcal{N}_{1,\eta}=\mathcal{H}_{1,\eta}$ and so $\mathrm{d_H}\left(\mathcal{N}_{1,\eta},\mathcal{N}_{1,0}\right)\stackrel{\eta\to 0^+}{\to} 0$, so that $\Xi_{\eta_k}\in\mathcal{N}_{1,\eta_k}$ for all k big enough, which contradicts that $\Xi_{\eta_k}\in\mathcal{N}_{2,\eta_k}$ for all k.

Thus, $\Xi_0\in\mathcal{N}_{3,0}\cup\mathcal{N}_{4,0}\cup\cdots\cup\mathcal{N}_{n,0}$ and so we can find $\Xi_{l_1}\in\boldsymbol{\Xi}_0$ with $\Xi_{l_1}\prec\Xi_0$ such that Ξ_{l_1} is not in $\mathcal{N}_{1,0}$. Let $(\Xi_{l_1,\eta_k})_{k\in\mathbb{N}}$ the sequence with $\Xi_{l_1,\eta_k}\in\boldsymbol{\Xi}_{\eta_k}$, for all k, such that $\mathrm{d_H}\left(\Xi_{l_1,\eta_k},\Xi_{l_1}\right)\stackrel{k\to\infty}{\to} 0$. By (HG) we have $\Xi_{l_1,\eta_k}\prec\Xi_{\eta_k}$ for all k big enough, but, as $\Xi_{\eta_k}\in\mathcal{N}_{2,\eta_k}$ for each k, then $\Xi_{l_1,\eta_k}\in\mathcal{N}_{1,\eta_k}$ for each k, but then we get that $\Xi_{l_1}\in\mathcal{N}_{1,0}$, which is a contradiction, so that the case $i=2$ is also proved.

Thus, by the second part in the proof of Theorem 5.2 we get $\eta_2\in(0,\eta_1]$ such that the sets $\mathcal{H}_{2,\eta}$, defined as in the previous theorem, are the sets $\mathcal{N}_{2,\eta}$ for $\eta\in(0,\eta_1]$, from which, in particular, $\mathrm{d_H}\left(\mathcal{N}_{2,\eta},\mathcal{N}_{2,0}\right)\stackrel{\eta\to 0^+}{\to} 0$.

For $i=3$, suppose $\Xi_0\notin\mathcal{N}_{3,0}$. Again, we then have that $\Xi_0\notin\mathcal{N}_{1,0}\cup\mathcal{N}_{2,0}=\mathcal{M}_{2,0}$, since as $\mathrm{d_H}\left(\mathcal{N}_{i,\eta},\mathcal{N}_{i,0}\right)\stackrel{\eta\to 0^+}{\to} 0$ for $i=1$ and 2, if $\Xi_0\in\mathcal{N}_{1,0}\cup\mathcal{N}_{2,0}$ we would have that $\Xi_{\eta_k}\in\mathcal{N}_{1,\eta_k}\cup\mathcal{N}_{2,\eta_k}$ for all k big enough, which contradicts that $\Xi_{\eta_k}\in\mathcal{N}_{3,\eta_k}$ for all k.

Thus, $\Xi_0\in\mathcal{N}_{4,0}\cup\cdots\cup\mathcal{N}_{n,0}$ and then we can find $\Xi_{l_1}\in\boldsymbol{\Xi}_0$ with $\Xi_{l_1}\prec\Xi_0$ such that $\Xi_{l_1}\notin\mathcal{M}_{2,0}$. As in the above cases, let $(\Xi_{l_1,\eta_k})_{k\in\mathbb{N}}$ the sequence with $\Xi_{l_1,\eta_k}\in\boldsymbol{\Xi}_0$, for all k, such that $\mathrm{d_H}\left(\Xi_{l_1,\eta_k},\Xi_{l_1}\right)\stackrel{k\to\infty}{\to} 0$. From (HG) we have that $\Xi_{l_1,\eta_k}\prec\Xi_{\eta_k}$ for all k big enough, but since $\Xi_{\eta_k}\in\mathcal{N}_{3,\eta_k}$ for each k, then $\Xi_{l_1,\eta_k}\in\mathcal{M}_{2,\eta_k}$ for each k, but then Ξ_{l_1} must be in $\mathcal{N}_{1,0}\cup\mathcal{N}_{2,0}=\mathcal{M}_{2,0}$, which contradicts the way it was chosen.

The argument must stop in a finite number of steps and so the proof is finished. $\square$

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(T. Caraballo and J. A. Langa) Departamento de Ecuaciones Diferenciales y Análisis Numérico, Universidad de Sevilla, Apdo. de Correos 1160, 41080-Sevilla Spain

E-mail address, T. Caraballo: `caraball@us.es`

E-mail address, J. A. Langa: `langa@us.es`

(A. N. Carvalho and E. R. Aragão-Costa) Instituto de Ciências Matemáticas e de Computaçao, Universidade de São Paulo-Campus de São Carlos, Caixa Postal 668, 13560-970 São Carlos SP, Brazil

E-mail address, E. R. Aragão–Costa: `ritis@icmc.usp.br`

E-mail address, A. N. Carvalho: `andcarva@icmc.usp.br`